European Journal of Mechanics A/Solids 29 (2010) 784–793

Contents lists available at ScienceDirect

European Journal of Mechanics A/Solids

journal homepage: www.elsevier.com/locate/ejmsol

Shakedown theorems and asymptotic behaviour of solids in non-smooth mechanics

Michaël Peigney

Laboratoire Central des Ponts et Chaussées, 48 boulevard Lefevbre, 75732 Paris cedex 15, France

ARTICLE INFO

Article history:
Received 15 December 2009
Accepted 31 March 2010
Available online 7 April 2010

Keywords:
Cyclic loading
Asymptotic behaviour
Non-smooth mechanics
Shape-memory alloys
Direct methods

ABSTRACT

Non-smooth mechanics is concerned with systems for which constraints are imposed on the physical quantities or their time derivatives. This article addresses the asymptotic behaviour (i.e. as time tends towards infinity) of such systems when they are submitted to a given loading history. A special emphasis is laid on shape-memory alloys structures, which are a typical example of systems for which an analysis in non-smooth mechanics is required. Extending the approach introduced by Koiter in plasticity, we state sufficient conditions for the energy dissipation to remain bounded in time, independently on the initial state. Concerning the asymptotic behaviour in the particular case of cyclic loadings, we also point out the fundamental differences that exist between the framework of plasticity and that of non-smooth mechanics.

1. Introduction

This article is concerned with the asymptotic behaviour (i.e. as time t tends towards infinity) of inelastic structures under prescribed loading histories. Much is known for elastic perfectly plastic structures: one of the earliest and most seminal contribution in that field has been made by Koiter (1960), following a pioneering idea of Melan (1936). The so-called Melan–Koiter static theorem gives a sufficient condition for the energy dissipation to remain bounded with respect to time. That situation is referred to as *shakedown*, and is associated with the intuitive idea that the structure behaves elastically for time t sufficiently large. The Melan–Koiter theorem has the distinctive property of being path-independent, i.e. independent on the initial state of the structure. In the particular case of cyclic loadings, it is also known (Halphen, 1978; Wesfreid, 1980) that the stress response $\boldsymbol{\sigma}(t)$ always converge towards a cyclic response $\boldsymbol{\sigma}_\infty(t)$ as $t \to +\infty$. Similarly, the rate of plastic strain $\dot{\boldsymbol{\alpha}}(t)$ converges towards a cyclic response $\dot{\boldsymbol{\alpha}}_\infty(t)$. Moreover, both $\boldsymbol{\sigma}_\infty(t)$ and $\dot{\boldsymbol{\alpha}}_\infty(t)$ have the same time period T as the applied loading. The plastic strain $\boldsymbol{\alpha}(t)$ does not necessarily converge towards a cyclic response, since $\int_0^T \dot{\boldsymbol{\alpha}}_\infty(t)\mathrm{d}t$ may be different from 0. That situation is referred to as *ratchetting* and implies the collapse of the structure through the accumulation of plastic strain. In the case where $\int_0^T \dot{\boldsymbol{\alpha}}_\infty(t)\mathrm{d}t = 0$, one classically distinguishes the cases of shakedown ($\dot{\boldsymbol{\alpha}}_\infty = 0$) and *accommodation* ($\dot{\boldsymbol{\alpha}}_\infty \neq 0$). In that last case, the plastic strain $\boldsymbol{\alpha}(t)$ converges towards a cyclic but non constant response $\boldsymbol{\alpha}_\infty(t)$. A crucial property of elastic perfectly plastic structures is that the asymptotic rate of plastic strains $\dot{\boldsymbol{\alpha}}_\infty$ is unique. This implies that the asymptotic regime (shakedown, accommodation, or ratchetting) is path-independent. That property has fostered the development of direct methods aiming at determining the asymptotic regime for a given cylic loading, without using a step-by-step incremental analysis (Zarka et al., 1988; Akel and Nguyen, 1989; Peigney and Stolz, 2001, 2003; Maitournam et al., 2002).

All the results mentioned so far apply for elastic perfectly plastic structures, and can be directly extended to the $C-$ class of generalized standard materials (Halphen and Nguyen, 1975). Outside of that framework, a lot of progress still remains to be made. Several attempts have been made to extend the Melan–Koiter theorem to various types of nonlinear behaviour (see (Pham, 2008) for an extensive review). However, as discussed in details by Pham (2008), some of the extensions proposed in the literature lead to non path-independent results which are therefore of little practical use. This is notably the case for shape-memory alloys: shakedown in shape-memory alloys structures has recently been studied by Feng and Sun (2007), but the shakedown theorem obtained by those authors has latter been recognized not to be path-independent (Pham, 2008). Wu et al. (1999) also have provided some results on shakedown in shape-memory alloys, but their study is limited to the local response of the material and does not take into account the equilibrium equations that would arise in the structural problem.

Shape-Memory Alloys (SMAs) display peculiar properties such as the superelastic behaviour or the shape-memory effect, which are both the result of a solid/solid phase transformation between

E-mail address: michael.peigney@polytechnique.org.

0997-7538/$ – see front matter © 2010 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.euromechsol.2010.03.011

different crystallographic structures (known as austenite and martensite). Much effort has been devoted to developing constitutive laws for describing the behaviour of SMAs (see e.g. the recent review by Kan and Kang (2010)). The phase transformation is typically described by an internal variable $\boldsymbol{\alpha}$ which – depending on the complexity of the material model – may be scalar or vectorial. A fundamental observation is that, in most of SMA models, the internal variable $\boldsymbol{\alpha}$ must comply with some a priori inequalities, resulting from the mass conservation in the phase transformation process. As a consequence, the internal variable $\boldsymbol{\alpha}$ is constrained to take values in a set $\mathcal{T}$ that is not a vectorial space. The presence of such constraints constitutes a crucial difference with plasticity models, and calls for special attention when the structural evolution problem is considered. This last point has been noted by Govindjee and Miehe (2001) in the context of numerical methods for simulating SMA structures: apart from few exceptions (Govindjee and Miehe, 2001; Peigney, 2006), most existing numerical methods handle the constraints in an ad hoc fashion, for lack of a consistent formulation of the time continuous evolution problem. It has to be observed, however, that mathematically consistent models of evolution problems in shape-memory alloys have been proposed (Frémond, 2002; Kružík et al., 2005). One possible approach is to resort to the so-called "non-smooth mechanics" framework (see (Frémond, 2002) and references therein), which is not restricted to shape-memory alloys and actually applies in the general situation where constraints are physically imposed on the state variables or their time-derivative. This article is devoted to studying the asymptotic behaviour of solids in such a framework.

The structure of the article is as follows: in Section 2 we describe the class of material models that is used in our study, introducing the relevant concepts of non-smooth mechanics. That class of materials is general enough to encompass standard models of plasticity and phase transformation. The corresponding structural evolution problem is presented in Section 3. Extending the reasoning of Koiter to non-smooth mechanics, we then proceed in Section 4 to give conditions ensuring that the energy dissipation remains bounded independently on time, whatever the initial state of the structure is. In a way similar to the original Melan–Koiter theorem in plasticity, the results of Section 4 deliver lower bounds on the domain of loadings for which shakedown occurs. An example problem is studied in Section 5. That example serves two purposes: firstly, it allows us to illustrate the shakedown theorems of Section 4 and study the optimality of the bounds delivered by those theorems. Secondly, for cyclic loadings, the problem of Section 5 allows us to show that the asymptotic behaviour of systems in non-smooth mechanics is fundamentally different – and actually more complex – than in plasticity: when the loading exceeds the shakedown limits predicted by the theorems of Section 4, the asymptotic regime is notably found to be strongly dependent on the initial state of the structure.

2. Constitutive laws

2.1. Unconstrained case

The local state of the material is described by the strain $\boldsymbol{\epsilon}$ and an internal variable $\boldsymbol{\alpha}$, living respectively in vectorial spaces denoted by $\mathbb{E}$ and $\mathbb{A}$. For now, we assume that $\boldsymbol{\alpha}$ is *unconstrained*, in the sense that $\boldsymbol{\alpha}$ is allowed to take any value in $\mathbb{A}$. The scalar products in $\mathbb{A}$ and $\mathbb{E}$ are denoted by . and :, respectively. The associated norms are denoted by $|\cdot|$ and $\|\cdot\|$, i.e. $|\boldsymbol{\alpha}| = \sqrt{\boldsymbol{\alpha}.\boldsymbol{\alpha}}$ for any $\boldsymbol{\alpha}\in\mathbb{A}$ and $\|\boldsymbol{\epsilon}\| = \sqrt{\boldsymbol{\epsilon}:\boldsymbol{\epsilon}}$ for any $\boldsymbol{\epsilon}\in\mathbb{E}$. Adopting the framework of generalized standard materials (Halphen and Nguyen, 1975), the behaviour of the material is determined by the free energy function $w:\mathbb{E}\times\mathbb{A}\to\mathbb{R}$ and the dissipation potential $\Phi:\mathbb{A}\to\mathbb{R}$. More precisely, denoting by $\dot{\boldsymbol{\alpha}}$ the left-time derivative of $\boldsymbol{\alpha}$, the constitutive equations are

$$\boldsymbol{\sigma} = \frac{\partial w}{\partial \boldsymbol{\epsilon}}(\boldsymbol{\epsilon},\boldsymbol{\alpha}), \quad \boldsymbol{A} = -\frac{\partial w}{\partial \boldsymbol{\alpha}}(\boldsymbol{\epsilon},\boldsymbol{\alpha}) \tag{1}$$

$$\boldsymbol{A}\in\partial\Phi(\dot{\boldsymbol{\alpha}}) \tag{2}$$

where $\boldsymbol{\sigma}$ is the stress, $\boldsymbol{A}$ is the thermodynamical force associated to $\boldsymbol{\alpha}$, and ∂ denotes the subdifferential operator. Recall that the subdifferential ∂f of a function $f:\mathbb{A}\to\mathbb{R}$ is the multi-valued mapping defined by

$$\partial f(x) = \left\{\tau\widehat{\mathrm{I}}\mathrm{A}\middle| f(y)-f(x)\geq\tau.(\boldsymbol{y}-\boldsymbol{x})\forall\boldsymbol{y}\widehat{\mathrm{I}}\mathrm{A}\right\} \tag{3}$$

If f is convex, then ∂f is a monotone operator (Brézis, 1972), i.e.:

$$(\boldsymbol{y}'-\boldsymbol{y})(\boldsymbol{x}'-\boldsymbol{x})\geq 0 \quad \text{for all } \boldsymbol{x}\in\mathbb{A},\boldsymbol{x}'\in\mathbb{A},\boldsymbol{y}\in\partial f(\boldsymbol{x}),\boldsymbol{y}'\in\partial f(\boldsymbol{x}') \tag{4}$$

In (2), the set $\partial\Phi(0)$ can be interpreted as the elasticity domain of the material, i.e. as the set of thermodynamical forces $\boldsymbol{A}$ compatible with a purely elastic behaviour ($\dot{\boldsymbol{\alpha}} = 0$).

In this article, we will consider free energy functions $w(\boldsymbol{\epsilon},\boldsymbol{\alpha})$ of the form

$$w(\boldsymbol{\epsilon},\boldsymbol{\alpha}) = \frac{1}{2}(\boldsymbol{\epsilon}-\boldsymbol{K}.\boldsymbol{\alpha}):\boldsymbol{L}:(\boldsymbol{\epsilon}-\boldsymbol{K}.\boldsymbol{\alpha})+f(\boldsymbol{\alpha}) \tag{5}$$

where $\boldsymbol{L}:\mathbb{E}\to\mathbb{E}$ is a symmetric positive linear mapping, $\boldsymbol{K}:\mathbb{A}\to\mathbb{E}$ is a linear mapping, and $f:\mathbb{A}\to\mathbb{R}$ is a positive differentiable function (not necessarily linear). The dissipation potential Φ will be assumed to satisfy the following properties:

$$\begin{array}{ll}\text{(i)} & \Phi \text{ is convex, positive, null at the origin}\\ \text{(ii)} & \exists r>0 \text{ such that } \{\boldsymbol{A}\in\mathbb{A}\,|\,|\boldsymbol{A}|\leq r\}\subset\partial\Phi(0)\end{array} \tag{6}$$

In the theory of generalized standard materials, the assumption (6)(i) is a usual requirement. Indeed, (6)(i) ensures the positiveness of the mechanical dissipation $\boldsymbol{A}.\dot{\boldsymbol{\alpha}}$, in accordance with the second law of thermodynamics. Using (3), (6)(i) is seen to imply that 0 is in the elasticity domain $\partial\Phi(0)$. The assumption (6)(ii) means that 0 is actually in the *interior* of $\partial\Phi(0)$. That additional assumption is not a stringent requirement and is verified for common material models.

With the form (5) of the free energy, the relation (1) becomes

$$\boldsymbol{\sigma} = \boldsymbol{L}:(\boldsymbol{\epsilon}-\boldsymbol{K}.\boldsymbol{\alpha}), \quad \boldsymbol{A} = {}^t\boldsymbol{K}:\boldsymbol{\sigma}-f'(\boldsymbol{\alpha}) \tag{7}$$

where ${}^t\boldsymbol{K}:\mathbb{E}\to\mathbb{A}$ is the transposed of $\boldsymbol{K}$, defined by $\boldsymbol{\alpha}.({}^t\boldsymbol{K}:\boldsymbol{\sigma}) = \boldsymbol{\sigma}:(\boldsymbol{K}.\boldsymbol{\alpha})$ for all $(\boldsymbol{\alpha},\boldsymbol{\sigma})\in\mathbb{A}\times\mathbb{E}$. The relation (7) shows that the total strain ϵ is the sum of an elastic strain $\boldsymbol{L}^{-1}:\boldsymbol{\sigma}$ and an inelastic strain $\boldsymbol{K}.\boldsymbol{\alpha}$. A wide range of commonly used plasticity models fall in the format (5)–(6). For latter reference, let us consider some specific examples.

Example 1. (*Uniaxial perfect plasticity*) The classical uniaxial model of elastic perfectly plastic materials corresponds to

$$\mathbb{E} = \mathbb{A} = \mathbb{R}, \; w(\epsilon,\alpha) = \frac{E}{2}(\epsilon-\alpha)^2, \; \Phi(\dot{\alpha}) = k|\dot{\alpha}| \tag{8}$$

where E and k are positive constants. The subdifferential $\partial\Phi(\dot{\alpha})$ is given by

$$\partial\Phi(\dot{\alpha}) = \begin{cases}[-k,k] & \text{if } \dot{\alpha} = 0\\ k & \text{if } \dot{\alpha}>0\\ -k & \text{if } \dot{\alpha}<0\end{cases} \tag{9}$$

The elasticity domain $\partial\Phi(0)$ of the material is the interval $[-k, k]$.

Example 2. (*Three-dimensional hardening plasticity*) In three-dimensional modelling of elastic plastic materials, the space $\mathbb{E}$ is generally taken as the space $\mathbb{R}^{3\times 3}_s$ of symmetric second-order tensors, and the corresponding scalar product is defined by