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Fundamental-solution-based finite element model for plane orthotropic elastic bodies

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ABSTRACT

A new hybrid finite element formulation is presented for solving two-dimensional orthotropic elasticity problems. A linear combination of fundamental solutions is used to approximate the intra-element displacement fields and conventional shape functions are employed to construct elementary boundary fields, which are independent of the intra-element fields. To establish a linkage between the two independent fields and produce the final displacement-force equations, a hybrid variational functional containing integrals along the elemental boundary only is developed. Results are presented for four numerical examples including a cantilever plate, a square plate under uniform tension, a plate with a circular hole, and a plate with a central crack, respectively, and are assessed by comparing them with solutions from ABAQUS and other available results.

1. Introduction

Orthotropic composite materials are now used extensively in the manufacturing of automobile parts and aerospace structures due to their high strength to weight and stiffness to weight ratios. The research to develop efficient numerical methods for accurately predicting the stress and failure behavior of structures containing orthotropic materials has attracted many research engineers and scientists (Jirousek and N'Diaye, 1990; Ochoa and Reddy, 1992).

In contrast to isotropic elastic material that has only two independent elastic constants, in orthotropic solids there are nine independent material constants for three-dimensional (3D) problems and four for two-dimensional problems (2D). The increase in the number of material constants means that solutions for orthotropic elastic problems are difficult to derive theoretically. As an alternative to analytical solutions and experiments, numerical simulations like the finite element method (FEM) and the boundary element method (BEM) play an important role in the process of designing and analyzing composite engineering structures, and much literature can be found in the field of orthotropic elastic materials including, for instance Jirousek and N'Diaye (1990), Huang et al. (2004), Pervez et al. (2005), Rao and Rahman (2005), Wang and Sun (2005), Asadpoure et al. (2006), Sladek et al. (2006), Sladek et al. (2007), Zhou et al. (2007), Ferreira et al. (2009), Danas and Ponte Castañeda (2009), as well as a book (Ochoa and Reddy, 1992) and the references therein.

However, as indicated by Qin (2000), Qin and Wang (2008) and Wang and Qin (2009), existing methods including conventional finite element formulation, the boundary element approach, meshless methods, and the hybrid Trefftz finite element method (HT-FEM) have some disadvantages in solving engineering structures with composite materials and local effects. For instance, in FEM it is necessary to evaluate time-consuming domain integrals and refined meshes near the local effects; moreover, conventional FEM may not guarantee satisfaction of the traction continuity condition on the common boundary of two adjacent elements. In contrast to the FEM, the BEM can reduce the computing dimensions by one, which may significantly reduce computing time. It is, however, time-consuming and tedious for the treatment of singular/supersingular integrals. Additionally, for multi-material problems, BEM requires extra equations to satisfy the interfacial continuity conditions. On the other hand, HT-FEM (Jirousek et al., 1995, Jirousek and Qin, 1996, Qin, 1995, 1996, 2003, 2004) inherits the advantages of FEM and BEM and can develop special elements for handling local effects. The drawbacks of HT-FEM are due to the construction of T-complete functions, the choice of truncated terms of T-complete functions, and the complex coordinate transformation required to keep the approach stable. Thus there is a need to develop new computational models that overcome those disadvantages. Through use of fundamental solutions rather than the T-complete functions in HT-FEM, a novel hybrid finite element formulation, called HFS-FEM, which is constructed using special fundamental solutions, was presented and successfully used for thermal analysis of a plate with special-purpose hole

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or fiber elements (Qin and Wang, 2008, Wang and Qin, 2009). In HFS-FEM, arbitrarily-shaped elements can be constructed by proper intra-element approximation with fundamental solutions, and source points are placed outside the element for removing the singularity of fundamental solutions. Combining the intra-element approximation, the independent element boundary interpolation and the new hybrid functional, the algorithm involves boundary integrals only. At the same time, the fundamental solutions used in HFS-FEM usually have simpler expression than the T-complete functions in HT-FEM, so that HFS-FEM discards the complicated coordinate transformation required in HT-FEM.

In contrast to the work in Qin and Wang (2008) and Wang and Qin (2009), this paper focuses on developing a fundamental-solution-based FEM for plane orthotropic elasticity. The formulation is based on a new hybrid variational functional and two groups of independent approximations to displacements which are defined within the element and on the element boundary, respectively. The Gaussian theorem is used to convert the domain integral appearing in the hybrid functional into the boundary integral, and the stationary condition of the hybrid functional is applied to produce the final solving equations and to establish the linkage of the assumed internal displacement field and boundary displacement field. Finally, several numerical results are presented to assess the performance of the proposed element formulation.

2. Basic equations in plane orthotropic elasticity

Let u_i, ε_{ij} and σ_{ij} be the components of displacement, strain and stress fields, respectively, with the subscripts i and j having the range (1,2). For homogeneous and orthotropic materials with two mutually orthogonal axes of elastic symmetry in the plane, the plane problem of classical elasticity is governed by the kinematic equations, the elastic constitutive expressions relating the in-plane stresses and strains, and the equilibrium equations without the body forces, that is (Lekhntiskii, 1963; Ting, 1996)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \partial_{,1} & 0 \\ 0 & \partial_{,2} \\ \partial_{,2} & \partial_{,1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{1}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \tilde{c}_{11} & \tilde{c}_{12} & 0 \\ \tilde{c}_{12} & \tilde{c}_{22} & 0 \\ 0 & 0 & \tilde{c}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} \tag{2}$$

$$\begin{bmatrix} \partial_{,1} & 0 & \partial_{,2} \\ 0 & \partial_{,2} & \partial_{,1} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = 0 \tag{3}$$

where $\gamma_{12} = 2\varepsilon_{12}$ denotes the engineering strain, the subscript comma represents the differential to the spatial coordinate component, i.e. $\partial_{,i} = \partial/\partial X_i$, $\partial_{,ij} = \partial^2/\partial X_i \partial X_j$, and

$$\tilde{c}_{11} = \frac{\tilde{s}_{22}}{\Lambda}, \quad \tilde{c}_{22} = \frac{\tilde{s}_{11}}{\Lambda}, \quad \tilde{c}_{12} = -\frac{\tilde{s}_{12}}{\Lambda}, \quad \tilde{c}_{66} = \frac{1}{\tilde{s}_{66}}$$

$$\Lambda = \tilde{s}_{11}\tilde{s}_{22} - \tilde{s}_{12}^2$$

with

$$\tilde{s}_{ij} = s_{ij} \quad (i,j = 1,2)$$

$$\tilde{s}_{66} = s_{66}$$

for the case of plane stress, and

$$\tilde{s}_{ij} = s_{ij} - s_{i3}s_{3j}/s_{33} \quad (i,j = 1,2)$$

$$\tilde{s}_{66} = s_{66}$$

for the case of plane strain. $s_{ij}(i,j = 1,2)$ and s_{13}, s_{23}, s_{66} are independent material compliance constants which can be expressed in terms of the engineering elastic constants.

In the material constants mentioned above, the subscripts 1 and 2 refer to the principal directions of material symmetry, which coincide here with the X_1 and X_2 reference axes.

Substituting Eqs. (1) and (2) into Eq. (3), we can obtain the following basic equations expressed in terms of displacement components u_i

$$\begin{bmatrix} \tilde{c}_{11}\partial_{,11} + \tilde{c}_{66}\partial_{,22} & \left(\tilde{c}_{12} + \tilde{c}_{66}\right)\partial_{,12} \\ \left(\tilde{c}_{12} + \tilde{c}_{66}\right)\partial_{,12} & \tilde{c}_{22}\partial_{,22} + \tilde{c}_{66}\partial_{,11} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0 \tag{4}$$

Moreover, appropriate boundary conditions should be complemented to keep the problems complete; that is, on the boundary of the domain of interest, we have

$$\begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \bar{t}_1 \\ \bar{t}_2 \end{Bmatrix} \tag{5}$$

or

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} \tag{6}$$

where overbar denotes specified values.

3. Fundamental solutions for plane orthotropic elasticity

The fundamental solution or Green's function plays an important role in the presented approach and is used to convert the domain integral into a boundary integral. It is necessary, therefore, to describe the fundamental solutions for plane orthotropic elasticity in order to provide a common source for reference in later sections.

In an orthotropic elastic infinite plane, for a unit force acting at $\mathbf{x}_s$ (source point), the corresponding singular fundamental solutions at a field point $\mathbf{x}$ are required to satisfy

$$\begin{bmatrix} \tilde{c}_{11}\partial_{,11} + \tilde{c}_{66}\partial_{,22} & \left(\tilde{c}_{12} + \tilde{c}_{66}\right)\partial_{,12} \\ \left(\tilde{c}_{12} + \tilde{c}_{66}\right)\partial_{,12} & \tilde{c}_{22}\partial_{,22} + \tilde{c}_{66}\partial_{,11} \end{bmatrix} \begin{Bmatrix} u_{1k}^* \\ u_{2k}^* \end{Bmatrix} + \{\delta_{k1}, \delta_{k2}\}^T \delta(\mathbf{x}, \mathbf{x}_s) = 0 \tag{7}$$

where the solutions u_{1k}^* and u_{2k}^* are given by (Rizzo and Shippy, 1970)

$$\begin{aligned} u_{11}^*(\mathbf{x}, \mathbf{x}_s) &= D\left(\sqrt{\lambda_1}A_2^2 \ln \rho_1 - \sqrt{\lambda_2}A_1^2 \ln \rho_2\right) \\ u_{12}^*(\mathbf{x}, \mathbf{x}_s) &= u_{21}^*(\mathbf{x}, \mathbf{x}_s) = DA_1A_2\left(\arctan\frac{r_2}{\sqrt{\lambda_2}r_1} - \arctan\frac{r_2}{\sqrt{\lambda_1}r_1}\right) \\ u_{22}^*(\mathbf{x}, \mathbf{x}_s) &= -D\left(\frac{A_1^2}{\sqrt{\lambda_1}}\ln \rho_1 - \frac{A_2^2}{\sqrt{\lambda_2}}\ln \rho_2\right) \end{aligned} \tag{8}$$

in which $\delta(\mathbf{x}, \mathbf{x}_s)$ denotes the Dirac delta function, u_{ik}^* are the induced displacement components in the i-direction at the field point $\mathbf{x}$ when a unit point force is applied along the k-direction at the source point $\mathbf{x}_s$, and