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Non-linear vibrations of imperfect free-edge circular plates and shells

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ABSTRACT

Large-amplitude, geometrically non-linear vibrations of free-edge circular plates with geometric imperfections are addressed in this work. The dynamic analog of the von Kármán equations for thin plates, with a stress-free initial deflection, is used to derive the imperfect plate equations of motion. An expansion onto the eigenmode basis of the perfect plate allows discretization of the equations of motion. The associated non-linear coupling coefficients for the imperfect plate with an arbitrary shape are analytically expressed as functions of the cubic coefficients of a perfect plate. The convergence of the numerical solutions are systematically addressed by comparisons with other models obtained for specific imperfections, showing that the method is accurate to handle shallow shells, which can be viewed as imperfect plate. Finally, comparisons with a real shell are shown, showing good agreement on eigenfrequencies and mode shapes. Frequency-response curves in the non-linear range are compared in a very peculiar regime displayed by the shell with a 1:1:2 internal resonance. An important improvement is obtained compared to a perfect spherical shell model, however some discrepancies subsist and are discussed.

1. Introduction

Geometric imperfections have been recognized since a long time for having a major effect on the linear and non-linear characteristics of thin-walled structures: from one structure to another one, even though manufactured by the same technique, it has been observed that eigenfrequencies and buckling loads can be different. In particular, a number of experimental and theoretical studies conducted in the 60–80s of the last century clearly establish that the initial deflection of thin structures such as plates and shells, that are unfortunately unavoidable when dealing with real structures, is a major cause for explaining the important discrepancies observed between theoretical results (calculated with an assumed perfect structure) and experimental observations (Donnell, 1976; Chia, 1980; Coppa, 1966; Tobias, 1951; Kubenko and Koval'chuk, 2004). Other important factors that could have been incriminated such as inaccuracy in the boundary conditions, inhomogeneity of the material or slight variations of the thickness, are not considered in this study and have been addressed in Chen et al. (2005), Gupta et al. (2007).

Among other thin shells, circular cylindrical shell with imperfections have been thoroughly studied, because of their wide importance in various engineering fields. The first investigations on the subject were generally limited to the effect of axisymmetric imperfections on the buckling loads (Koiter, 1963; Rosen and Singer, 1974). Asymmetric imperfections were then introduced in Rosen and Singer (1976). Geometrically non-linear, large-amplitude vibrations are considered in Gonçalves (1994). Recent papers give overview of the numerous results available for cylindrical shells, where forced and parametric excitation, flutter, experimental measurements of imperfections and fitting to theoretical models, are detailed, see Amabili (2003), Amabili and Païdoussis (2003), Kubenko and Koval'chuk (2004) and references therein.

The influence of imperfections on the behaviour of plates has also been reported by many investigations. Rectangular plates are generally treated for their wide use in practice, as well as for the ease of using Cartesian co-ordinates. Free vibrations with large amplitude are treated in Celep (1976). Quantitative results on the effect of an imperfection on eigenfrequencies and buckling loads are given in Hui and Leissa (1983). The type of non-linearity (hardening or softening behaviour of non-linear oscillations) is also addressed by Hui (1984). These two studies clearly establish that large deviations from the perfect theory are present, for amplitude of imperfections being only a fraction of the plate thickness. However, all the presented results are obtained via a crude approximation consisting in keeping only one mode in the Galerkin expansion, so that some of their results must be reconsidered with more accurate expansions. Forced vibrations with experimental results are shown in Yamaki and Chiba (1983), Yamaki et al. (1983). The transition from the hardening behaviour of flat plates

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to the softening behaviour of imperfect rectangular plates is also addressed in Lin and Chen (1989), as well as in Ostiguy and Sassi (1992), where the response to simultaneous forced and parametric excitation is investigated.

The case of circular plates has received less attention. Hui, with a single-mode expansion and axisymmetric restriction, studied the type of non-linearity with various (mainly clamped and simply supported) boundary conditions (Hui, 1983). Yamaki et al., with a three-mode expansion and also with an axisymmetric restriction, studied both theoretically and experimentally the forced response of clamped circular plates (Yamaki et al., 1981a, 1981b).

In all the precedent studies, the method used for analyzing the results is in most of the cases a Galerkin expansion, based either on ad-hoc basis functions, or on the eigenmodes of the perfect structure. However, a number of them used a single-mode expansion, see e.g. Celep (1976), Hui and Leissa (1983), Hui (1984), Lin and Chen (1989), Hui (1983). As precised by a number of studies (Yamaki and Chiba, 1983; Yamaki et al., 1981a; Ilanko, 2002), these truncations are too severe and may lead to incorrect results, especially when dealing with non-linear vibrations. More particularly, it has been demonstrated in Nayfeh et al. (1992), Touzé et al. (2004) that, when predicting the type of non-linearity (hardening/softening behaviour) of a structure with an initial curvature, single-mode truncation leads to erroneous results.

The aim of the present study is thus to reconsider some of the precedent results on imperfect plates, while specifically overstepping the limitations underlined in the current state-of-the-art. More particularly, the following points are addressed. Firstly, the axisymmetric restriction for the case of circular plates is not retained. Secondly, free-edge boundary conditions, that are generally not treated in the literature, are considered. Thirdly, a Galerkin expansion using an arbitrary number of expansion functions is used, hence overstepping the usual limitation to a one-mode expansion. The initial shape of the structure as well as its deflection in vibrations are expanded on the same expansion functions, the mode shapes of a circular plate. It leads to analytical expressions of the coupling coefficients, as functions of the non-linear cubic coefficients of the perfect plate. The convergence of the numerical solutions is systematically addressed by comparing the obtained results with the spherical shallow shell model developed in Thomas et al. (2005b), as well as with finite-elements solutions. It is shown that converged solutions are available with a reasonable number of expansion functions, for amplitudes of imperfections up to 30 times the thickness of the plate. This allows considering a shallow shell model directly from a plate model.

Finally, comparisons with experimental results on a real shell are reported. Numerical problems related to the approximation of the measured geometry are discussed. The linear results provided by the imperfect plate model are compared to measurements, showing an important improvement with comparison to the predictions brought by a perfect shallow shell model. At the non-linear level, frequency-response curves are drawn, in the specific regime obtained when forcing the first axisymmetric mode, the eigenfrequency of which is twice those of the two companion modes with six nodal diameters. The complete experimental report has already been addressed in Thomas et al. (2007), showing that the non-linear terms predicted by a perfect spherical shell model are very far from the measured ones. Although showing a better agreement with experiment, some discrepancies subsist in some non-linear coefficients, giving an incorrect prediction of the instability regions. Finally the complete model predicts the correct type of non-linearity of the shell, but with an enhanced non-linearity.

2. Theoretical formulation

2.1. General case

2.1.1. Local equation

A thin plate of diameter $2a$ and thickness h (with $h \ll a$), made of a homogeneous isotropic material of density ρ, Poisson's ratio ν and Young's modulus E, is considered. The equations of motion for perfect circular plates subjected to large deflections, moderate rotations and with small strain, used in the sequel, are known as the dynamic analogues of the von Kármán equations, where damping and forcing have been added. In-plane and rotatory inertia are neglected so that an Airy stress function F is used. Hence, at a given point of co-ordinates (r,θ), the equations of motion are given in terms of the Airy stress function F and the transverse displacement w along the normal to the mid-surface of the plate, for all time t:

$$D\Delta\Delta w + \rho h \ddot{w} = L(w,F) - c\dot{w} + p, \tag{1a}$$

$$\Delta\Delta F = -\frac{Eh}{2} L(w,w), \tag{1b}$$

where

$$L(w,F) = w_{,rr}\left(\frac{F_{,r}}{r} + \frac{F_{,\theta\theta}}{r^2}\right) + F_{,rr}\left(\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2}\right) - 2\left(\frac{w_{,r\theta}}{r} - \frac{w_{,\theta}}{r^2}\right)\left(\frac{F_{,r\theta}}{r} - \frac{F_{,\theta}}{r^2}\right), \tag{2}$$

$D = Eh^3/12(1-\nu^2)$ is the flexural rigidity, c is a viscous damping coefficient and p represents the external load normal to the surface of the plate. $(\ddot{\ })$ denotes a twice differentiation with respect to time t and $(\cdot)_{,\alpha\beta}$ is the partial derivative with respect to α and β. The expression of the Airy stress function F as a function of the membrane stresses can be found in Touzé et al. (2002). Laplacian operator writes:

$$\Delta(\cdot) = (\cdot)_{,rr} + \frac{1}{r}(\cdot)_{,r} + \frac{1}{r^2}(\cdot)_{,\theta\theta}. \tag{3}$$

As shown in Fig. 1, the geometric imperfections are included in the formulation by splitting the transverse displacement w into a static part w_0 and a dynamic part $\tilde{w}$, so that:

$$w(r,\theta,t) = \tilde{w}(r,\theta,t) + w_0(r,\theta). \tag{4}$$

In order to satisfy the static equilibrium initial state, both p and F are similarly split in two quantities, a time-dependent one and a static one:

$$F = \tilde{F} + F_0, \tag{5a}$$

$$p = \tilde{p} + p_0. \tag{5b}$$

Substituting Eqs. (5) in Eqs. (1), one obtains:

$$D\Delta\Delta\tilde{w} + D\Delta\Delta w_0 + \rho h\ddot{\tilde{w}} = L(\tilde{w},\tilde{F}) + L(w_0,\tilde{F}) + L(\tilde{w},F_0) + L(w_0,F_0) - c\dot{\tilde{w}} + \tilde{p} + p_0, \tag{6a}$$

$$\Delta\Delta\tilde{F} + \Delta\Delta F_0 = -\frac{Eh}{2}\left[L(\tilde{w},\tilde{w}) + 2L(\tilde{w},w_0) + L(w_0,w_0)\right]. \tag{6b}$$

The static equilibrium leads to the following relationships:

$$D\Delta\Delta w_0 = L(w_0,F_0) + p_0, \tag{7a}$$

$$\Delta\Delta F_0 = -\frac{Eh}{2} L(w_0,w_0). \tag{7b}$$

Since a purely geometric imperfection, without initial in-plane stress, is considered in this study, the static membrane stress term