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A statistically consistent fatigue damage model based on Miner's rule

Qin Sun, Hong-Na Dui *, Xue-Ling Fan

School of Aeronautics, Northwestern Polytechnical University, Xi'an 710072, PR China

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ABSTRACT

With Monte Carlo sampling method, a statistically consistent fatigue damage model under constant and variable amplitude loadings based on linear Miner's rule is proposed, which can quantitatively depict the probabilistic properties of fatigue damage and life. Numerical simulation shows that linear Miner's damage criterion is statistically inconsistent; through some modification from a probabilistic point of view, a statistically consistent damage criterion is first established. To validate the statistical model, numerical verification of a supposed two-level cyclic loading and experimental verification of fatigue tests for Al-alloy straight lugs available in the literature are successfully conducted. The model predictions coincide quite well with both engineering hypotheses and experimental observations, compared with Miner's model, indicating that the model can be regarded quantitatively accurate for engineering application.

1. Introduction

The simplification of complex fatigue loading spectra has significant impact on the experimental speeding up of endurance verification for mechanical components, in which the most challenging work is to predict fatigue damage and life under variable amplitude loading conditions through more accurate damage models, especially for its probabilistic properties. The fatigue progress under service loading is uncertain in nature, due to various sources of uncertainty such as material properties, external applied loading, notch geometries, defects.

Over the past several decades, a variety of cumulative damage theories [1,2] have been developed for materials. The LDR (linear damage accumulation rule), also known as Miner's rule [3], is the simplest and most popular one; however, it has two main disadvantages. First, it is a linear cumulative damage rule, without taking load sequence effect into account; second, it is a deterministic damage model, which means it cannot account for the statistical dispersion of cumulative damage. Dozens of alternative theories have been proposed to take the place of Miner's rule, such as nonlinear cumulative damage rule [4,5], damage curve approach (DCA) [6], approaches based on crack growth [7], energy-based damage theories [8] and continuum damage mechanics approaches [9,10]. In the authors' knowledge, almost all these damage models are deterministic and most of nonlinear damage theories need lots of computational effort and require detailed information such as material parameters, crack geometry, crack growth laws and other mechanisms. However, in engineering applications, the information is usually not fully available. Because Miner's rule is simple to apply and seems to give not so much worse results than the others, it has remained widespread engineering application. Besides, many experimental analyses [11] have shown that Miner's rule can predict the mean value of fatigue life under random loading for engineering structure at some accurate degree.

In recent decades, many probabilistic approaches have been proposed to describe the statistics of fatigue life and damage under constant and variable amplitude loadings. Birnbaum et al. [12] introduced a statistical interpretation of Miner's rule from a probabilistic point of view. Shimokawa et al. [13] used p–S–N curves and Miner's rule in statistical terms to analyze the fatigue reliability for two-step fatigue tests, with both lognormal and Weibull distribution assumption of fatigue life. Rowatt et al. [14] employed p–S–N curves and Markov chain models for life prediction of composite laminates. Pascual et al. [15] proposed a random fatigue-limit model to describe the scatter in S–N curves obtained by constant amplitude fatigue tests. Shen et al. [16] used lognormal distribution and Miner's rule for fatigue life prediction under a narrow-band Gaussian stochastic stress process.

In this paper, assuming fatigue life is distributed as Weibull distribution, a statistically consistent damage criterion is first formulated based on linear Miner's rule $\sum 1/N = 1$ from a probabilistic perspective, where a consistent index is introduced. Then, through some transformation, a statistically consistent fatigue damage model for constant amplitude loading has been proposed, which is next extended to the model under variable amplitude loading.

As for the critical damage Dc, i.e. the damage value at failure, there are mainly two points of view: one, Dc is deterministic, equal

* Corresponding author. Present address: Aeronautics Building A517, School of Aeronautics, Northwestern Polytechnical University, Laodong Road, Xi'an, Shaanxi 710072, China. Tel./fax: +86 02988492850.

E-mail address: duihn060379@126.com (H.-N. Dui).

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to or smaller than unit; the other, Dc is a random variable, with mean value equal to unit. In the paper, Dc is assumed as a random variable in the derivation of a statistically consistent damage criterion, while it is taken as unit in the statistical damage model transformed by the previous damage criterion.

In addition, the service loading of mechanical components is commonly random in practice, and among various cycle counting techniques, rain flow counting method is mainly used. In this paper, only constant amplitude loading and multi-level cyclic loading are applied, thus the rain flow counting method is unnecessary. However, it is still introduced here for the generality of practical applications of the statistical fatigue damage model.

2. A statistically consistent fatigue damage model based on Miner's rule

2.1. Weibull distribution

Weibull distribution is one of the most widely used lifetime distributions in reliability engineering, which is versatile to take on the characteristics of other types of distributions. And it is generally assumed that fatigue life of material follows two-parameter Weibull distribution. For the next statistical analysis of fatigue model, related knowledge of Weibull distribution is firstly required, such as Monte Carlo sampling [17], random sampling tests [17,18], parameter estimation [19,20] and goodness of fit tests [21,22].

Assuming $X–W(\alpha, \beta)$, its probability density function (PDF) and cumulative distribution function (*CDF*) can be written as:

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right), \quad x \geqslant 0 \tag{1}$$

$$F(x) = 1 - \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) \tag{2}$$

where α and β are the shape and scale parameters, respectively.

The mean value μ and standard deviation σ can be obtained according to the following equation.

$$\mu = \beta \cdot \Gamma\left(1 + \frac{1}{\alpha}\right), \quad \sigma = \beta \cdot \left(\Gamma\left(1 + \frac{2}{\alpha}\right) - \left(\Gamma\left(1 + \frac{1}{\alpha}\right)\right)^2\right)^{\frac{1}{2}} \tag{3}$$

It is common to assume that shape parameter α of fatigue life is determined by material properties and scale parameter β depends on applied loadings. Through lots of experimental analyses, it has been shown that the modal number of shape parameter can be taken as 4 and 1.25 for metallic and composite materials, respectively [23].

2.2. A statistically consistent fatigue damage criterion

Linear Miner's damage criterion is widely used in engineering, which for constant amplitude loading takes the form of

$$\sum_{i=1}^{n} \frac{1}{N_i} = 1 \tag{4}$$

where N_i is the fatigue life of some material under some stress level, which actually obeys some distribution; n is the number of load cycles in fatigue test; and the critical damage value is assumed to be 1.

Obviously, when N is taken as a constant, such as the mean value of life distribution, Eq. (4) can be simplified to the form of $n/N = 1$, which is the most common Miner's damage criterion and the equation is always true when $n = N$. However, in fact, N has probability characteristics, thus n is correspondingly statistical rather than deterministic.

With a given distribution of N, the equation can be called statistically consistent if n obtained from Eq. (4) with Monte Carlo sampling method is approximate to N with respect to probability distribution. In practice, fatigue life N is exactly reflected in the number of load cycles to failure, so it is necessary that the damage criterion be statistically consistent, however, it can be verified that Eq. (4) is statistically inconsistent as follows.

Assuming $N–W(\alpha, \beta)$, $\alpha = 4$, $\beta = 10{,}000$ and accordingly $\mu = 9064$, $\sigma = 2542.9$ based on Eq. (3), through sufficient random samplings of Eq. (4), here 1000 times is enough, 1000 different n can be obtained and then statistics of n are calculated: $\mu = 8150$, $\sigma = 37$. Note that both the mean value and standard deviation are smaller than that of original N, especially for standard derivation. Thus it can be concluded that Eq. (4) is statistically inconsistent.

In order to make Miner's damage criterion statistically consistent, some probabilistic modification must be added to the original form. To increase the mean value and standard deviation of n, introducing a consistent index a (greater than 1) and a random disturbance Δ to the left and right sides of Eq. (4) respectively, will result in

$$\sum_{i=1}^{n} \left(\frac{1}{N_i}\right)^{a} = 1 + \Delta, \quad \Delta = \frac{N_i - \mu}{\mu} \tag{5}$$

where N_i is a random fatigue life from some distribution, and μ is the mean value of the life distribution.

In Eq. (5), variable work absorption per cycle is defined as $(1/N_i)^a$. The right-hand side term $(1 + \Delta)$ is equivalent to the assumption that the critical damage is a random variable. Assuming $N–W(\alpha, \beta)$, $(1 + \Delta)$ can be described by $W(\alpha, \beta/\mu)$ distribution, with mean value and standard derivation equal to 1 and σ/μ, respectively, where σ is the standard derivation of fatigue life N.

Assuming $N–W(\alpha, \beta)$, $\alpha = 4$, $\beta = 10{,}000$, through large number of Monte Carlo sampling tests, it is found that when consistent index a is taken as 1.011, the mean values of n_f and N can be comparable. Repeating Monte Carlo sampling of Eq. (5) for 1000 times, the statistics of n can be obtained for comparison with original fatigue life, including the maximum likelihood estimations of two parameters, the mean value and standard derivation. In order to verify the generality of Eq. (5), another Weibull distribution with greater skewness (skewness is a measure of the asymmetry of the probability distribution, and greater skewness indicates that the tail on the right side is much longer than the left side.) is assumed, $\alpha = 2$, $\beta = 10{,}000$ and correspondingly, $\mu = 8862$, $\sigma = 4632.5$, $a = 1.054$ (determined by Monte Carlo sampling test). In Table 1, there list statistics of n corresponding to the two assumed fatigue lives, $\alpha = 4$ and $\alpha = 2$. In addition, the schematic comparisons of PDF between N and n are illustrated in Fig. 1.

It can be noted that n is close to N with respect to the mean value and standard deviation, as well as the maximum likelihood estimations of parameters α and β. Furthermore, in the goodness of fit test with the original Weibull distribution, n has passed K–S test and W^2 test quite well. In addition, as shown in Fig. 1, the schematic PDFs of N and n are quite close. Therefore, the new cumulative damage criterion expressed in Eq. (5) can be considered statistically consistent.

The consistent index a is closely related to statistical parameters α and β of fatigue life distribution and can be solved by Monte

Table 1
The statistics of n corresponding to two assumed fatigue lives.

The original N	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\mu}$	$\hat{\sigma}$
$\alpha = 4$	3.9	9980	9033	2592.8
$\alpha = 2$	2	9930	8800	4600