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Prediction of non-proportionality factors of multiaxial histories using the Moment Of Inertia method

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ABSTRACT

This work studies further an approach originally proposed to evaluate equivalent stress and strain ranges in non-proportional (NP) load histories, called the Moment Of Inertia (MOI) method. The MOI method assumes that the path contour in the deviatoric stress or strain diagram is a homogeneous wire with unit mass. The center of mass of such wire gives then the mean component of the path, while the moments of inertia of the wire can be used to obtain the equivalent stress or strain ranges. The MOI method is an alternative to convex enclosure methods, such as Dang Van's Minimum Ball or the Maximum Prismatic Hull methods, without the need for computationally-intensive search algorithms or adjustable parameters. The MOI method is extended here to calculate as well the non-proportionality factor F_{np} of generic multiaxial load histories, formulated in an alternative sub-space of the deviatoric plastic strains. Experimental results for 14 different multiaxial histories prove the effectiveness of the MOI method to predict the observed non-proportionality factors. Hence, it can be a most useful tool for the computation of multiaxial fatigue damage in practical applications.

1. Introduction

Several multiaxial fatigue damage models have been introduced in the literature, such as the ones proposed by Sines, Crossland, Findley, McDiarmid, Brown–Miller, Fatemi–Socie and Smith–Watson–Topper (SWT) [1]. All of them require some measure of an equivalent stress or strain range, which may be difficult to obtain for non-proportional (NP) multiaxial load histories.

For a given multiaxial stress–strain NP history, the fatigue damage can be calculated by projecting the history onto a candidate plane at the critical point [1]. This critical plane approach is simple to compute for Case A cracks, which initiate perpendicular to the free surface. In this case, the in-plane shear stress or strain may be counted using a uniaxial rainflow algorithm [2]. On the other hand, for Case B cracks, which initiate at a 45° angle from the free surface, a multiaxial rainflow count must be performed to identify individual cycles formed by the in-plane and out-of-plane shear components [3].

For each rainflow-counted cycle, the equivalent stress or strain range is often computed using the so-called convex enclosure methods [4], which try to find circles, ellipses or rectangles that contain the entire projected path in the 2D case, or hyperspheres, hyperellipsoids or hyperprisms in a generic 5-dimensional (5D) equivalent stress space. The traditional convex enclosure methods have been reviewed in [4]: the Minimum Ball, Minimum Circumscribed Ellipsoid, Minimum Volume Ellipsoid, Minimum F-norm Ellipsoid (MFE), Maximum Prismatic Hull and Maximum Volume Prismatic Hull. These methods make use of stress and strain parameters such as the von Mises stress and strain ranges $\Delta\sigma_{Mises}$ and $\Delta\varepsilon_{Mises}$, defined by:

$$\Delta\sigma_{Mises} = \frac{\sqrt{(\Delta\sigma_x - \Delta\sigma_y)^2 + (\Delta\sigma_x - \Delta\sigma_z)^2 + (\Delta\sigma_y - \Delta\sigma_z)^2 + 6(\Delta\tau_{xy}^2 + \Delta\tau_{xz}^2 + \Delta\tau_{yz}^2)}}{\sqrt{2}} \quad (1)$$

$$\Delta\varepsilon_{Mises} = \frac{\sqrt{(\Delta\varepsilon_x - \Delta\varepsilon_y)^2 + (\Delta\varepsilon_x - \Delta\varepsilon_z)^2 + (\Delta\varepsilon_y - \Delta\varepsilon_z)^2 + 1.5(\Delta\gamma_{xy}^2 + \Delta\gamma_{xz}^2 + \Delta\gamma_{yz}^2)}}{\sqrt{2}\cdot(1+\bar{\nu})} \quad (2)$$

where the $\bar{\nu}$ is the mean (or effective) Poisson coefficient $\bar{\nu} = (0.5\varepsilon_p + \nu_e\varepsilon_e)/(\varepsilon_p + \varepsilon_e)$, while ε_e and ε_p are the elastic and plastic components of the strains, and ν_e and ν_p are the elastic and plastic Poisson coefficients ($\nu_p = 0.5$ assuming plastic strains conserve material volume).

Extensive simulations from [4] showed that all convex enclosure methods can lead to poor predictions of the *mean* stresses or strains, if they are assumed as located at the center of the ball, ellipse or prism, as seen in Fig. 1(a), which shows a stress path shaped very differently from an ellipse and its Minimum F-norm Ellipsoid (MFE) enclosure. Convex enclosure methods may also

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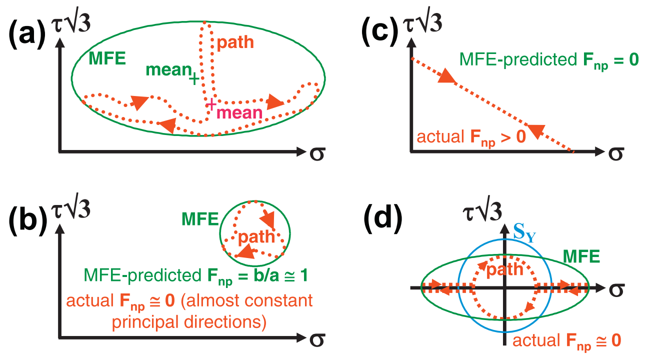

Fig. 1. History path examples showing the inadequacy of convex enclosure methods, such as the Minimum F-norm Ellipsoid (MFE), to predict mean components or the non-proportionality factor F_{np}.

result in poor estimates of stress or strain *amplitudes*, in special for highly non-convex NP history paths, such as cross or star-shaped paths.

If only the stress or strain history is measured, then an incremental plasticity algorithm must be implemented to obtain the stress–strain behavior caused by a NP loading. To account for NP hardening effects, it is necessary to correctly evaluate the non-proportionality factor F_{np} associated with the load history and the additional hardening coefficient α_{np}. The factor F_{np} depends solely on the shape of the history path [5], while α_{np} depends not only on the material and its microstructure, but also on the strain amplitudes involved in the history. The additional hardening coefficient can be estimated from

$$\alpha_{np} = \frac{\sigma_{OP}}{\sigma_{IP}} - 1 \tag{3}$$

where σ_{IP} and σ_{OP} are the equivalent von Mises stress amplitudes obtained under the same strain level for, respectively, in-phase ($F_{np} = 0$) and 90° out-of-phase ($F_{np} = 1$) loadings. This σ_{OP}/σ_{IP} ratio is usually calculated at high plastic strains, however it can be defined at any strain level, resulting in some strain amplitude dependence of α_{np}.

If α_{np} is eliminated from the F_{np} equation, then F_{np} can be obtained for a given von Mises stress amplitude σ from

$$F_{np} = \frac{(\sigma/\sigma_{IP}) - 1}{(\sigma_{OP}/\sigma_{IP}) - 1} \tag{4}$$

as long as σ is measured in the same material and under a similar strain level as the one from the σ_{IP} and σ_{OP} measurements. Using the above equation, F_{np} can be calculated from experiments without the need to explicitly obtain α_{np} or to worry about its strain amplitude dependence. In the absence of experimental data to measure σ, σ_{IP} and σ_{OP}, the NP factor F_{np} must be estimated from the load history path. The main F_{np} estimates are presented next.

2. Estimates of the non-proportionality factor F_{np}

Originally, F_{np} was estimated from the aspect ratio of the convex enclosure that contains the history path (e.g. the aspect ratio b/a of an enclosing ellipse with semi-axes a and b). But such convex enclosure estimates can lead to poor predictions of F_{np}, as seen in Fig. 1(b). This example shows a path that does not encircle the origin of the von Mises $\sigma \times \tau\sqrt{3}$ diagram, while entirely located far away from it. Despite the almost circular shape of the enclosing Minimum F-norm Ellipsoid (MFE), which would suggest $F_{np} \cong 1$, the principal direction in fact varies very little along such path, since the angle between each point in the path and the origin of the 2D diagram varies very little during each cycle – thus, the actual F_{np} should be very small in this example.

Another notable example where convex enclosures fail to calculate F_{np} is shown in Fig. 1(c), where a loading path describes a straight line that does not cross the origin of the diagram. This particular path induces a 45° variation of the principal direction, implying in F_{np} 0, however any convex enclosure method would predict $F_{np} = 0$ for such straight line. This path is an interesting example of how an in-phase loading (which is represented by a straight path) can be non-proportional (making the principal direction vary). Note also that convex enclosure methods can lead to poor F_{np} predictions even in paths that encircle the origin, in special when the path shape is very different from an ellipse or rectangle, or when the mean value of the path is not located close to the origin.

The use of the stress path to estimate F_{np} is also questionable. Fig. 1(d) shows a stress path that combines a purely elastic tension–torsion portion (well inside the yield surface with radius S_Y) with uniaxial tension–compression plastic straining. Since NP hardening is caused by plastic straining, the purely elastic portion should not influence the value of F_{np}. As plastic strains only occur along such path under uniaxial conditions, it is expected that $F_{np} = 0$, which is confirmed by experiments and incremental plasticity simulations using Tanaka's NP model [6]. However, a convex enclosure method applied to such stress path would wrongfully predict F_{np} much greater than zero, as suggested by the MFE ellipse in Fig. 1(d). Therefore, any accurate F_{np} estimation method should be based on the plastic strain path, not on the stress or total strain path.

Several methods have been proposed to estimate F_{np}, besides the ones based on convex enclosures. Kanazawa et al. [7] estimated F_{np} as a rotation factor, defined by the ratio between the shear strain range at 45° from the maximum shear plane and the maximum shear strain range. This factor correctly tends to the limits $F_{np} = 0$ for proportional loadings and $F_{np} = 1$ for 90° out-of-phase strain histories (assuming the relation $\gamma_a = (1 + \bar{\nu}) \cdot \varepsilon_a$ between strain amplitudes for Case A cracks [1]). But it fails to correctly compute F_{np} for more complex histories.

Itoh et al. [8] estimated F_{np} using an integral definition along the strain path:

$$F_{np} = \frac{\pi}{2T\varepsilon_{Imax}} \int_0^T \varepsilon_I(t) \cdot |\sin \xi(t)| \cdot dt \tag{5}$$

where $\varepsilon_I(t)$ is the absolute value of the maximum principal strain at each instant t, ε_{Imax} is the maximum value of $\varepsilon_I(t)$ along the entire path, $\xi(t)$ is the angle between the principal directions associated with $\varepsilon_I(t)$ and ε_{Imax}, and T is the time period of the path.

Itoh's method works for simple 2D (e.g. tension–torsion) histories, but it should not be applied to more general 3D to 6D histories, since it is based on a scalar measure, the angle $\xi(t)$. For instance, if the directions of $\varepsilon_I(t)$ along a load path describe a cone with symmetry axis in the direction of ε_{Imax}, then $\xi(t)$ would be constant and equal to half the cone apex angle, regardless of the chosen path. Constant amplitude or 90° out-of-phase cycles could result in the same $\xi(t)$ and $\varepsilon_I(t)$ histories, wrongfully calculating the same F_{np} for both cases. Instead of using the scalar measure $\xi(t)$, the direction of $\varepsilon_I(t)$ would need to be defined by a vector of at least two elements to be able to distinguish between these example paths.

To calculate F_{np} of a more general 6D load path, Bishop [9] introduced a 6×6 inertia tensor termed the Rectangular Moment Of Inertia (RMOI) of the stress path, which can be expressed using Voigt-Mandel's stress representation $\bar{\sigma} \equiv \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy}\sqrt{2} & \tau_{xz}\sqrt{2} & \tau_{yz}\sqrt{2} \end{bmatrix}^T$ by

$$I_\sigma \equiv \frac{1}{p_\sigma} \cdot \oint (\bar{\sigma} - \bar{\sigma}_m) \cdot (\bar{\sigma} - \bar{\sigma}_m)^T \cdot |d\bar{\sigma}| \tag{6}$$

where the mean component $\bar{\sigma}_m$ and accumulated stress p_σ are also integrated along the stress path, calculated from