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Shakedown limit theorems for frictional contact on a linear elastic body

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ABSTRACT

The paper considers the problem of a body composed of a linear elastic material in contact with a planar rigid surface with an inter-surface coefficient of Coulomb friction μ. The body is subjected to a cyclic history of loading, $\lambda P_i(x_i,t)$ where λ denotes a scalar multiplier. The objective is to assess the conditions when movement occurs between the elastic body and the surface. The problem has a close analogy with classical plasticity, where shakedown and limit load bounds exist. However, existing plasticity theory is not generally applicable to frictional slip as it obeys a non-associated flow rule. In this paper upper and lower bound shakedown theorems are derived in terms of the Coulomb coefficient of friction μ. It is shown that optimal kinematic and static bounds do not coincide. This implies that for a prescribed $\lambda P_i(x_i,t)$ there are ranges of μ for which shakedown definitely occurs and for which shakedown definitely does not occur, independent of the state of slip at the beginning and end of the cycle. However there exists an intermediate range of μ for which it is not possible to say whether shakedown or ratchetting occurs without detailed knowledge of the slip displacements at the beginning and end of the cycle of loading. This observation accords with simulations reported by Flecek R.C., Hills D.A., Barber J.B. and Dini D., (2015).

A programming method for the shakedown limit is developed, based on the Linear Matching Method. The method is illustrated by a simple example. The theory derived in this paper paves the way for a new theory of limit and shakedown analysis for structures and materials with a non-associated flow rule.

1. Introduction

The paper considers the problem of a body, composed of a linear elastic material, in contact with a planar rigid surface with an inter-surface coefficient of Coulomb friction μ as shown schematically in Fig. 1. The body is subjected to a cyclic history of loading, $\lambda P_i(x_i,t)$ where λ denotes a scalar multiplier. The objective is to assess the conditions when cyclic slip does and does not occur between the elastic body and the surface. Surface slip problems may be regarded as a subclass of the general shakedown problem where the slip surface may be regarded as an infinitely thin layer of a plastic material with a yield condition in the form of a slip condition. A state of shakedown in a body, that exhibits both linear elastic behaviour and non-linear inelastic slip, is the existence of cyclic states where prior slip results in "protective" residual stresses. These residual stresses, combined with a varying linear elastic stresses, gives rise to purely elastic behaviour over a cycle of loading, i.e. the surface slip condition is not violated. The shakedown limit corresponds to the extreme loading case for which such a state can exist.

In classical shakedown theory two theorems exist, Melan's lower bound equilibrium theorem (Melan, 1938; Koiter, 1960), and the upper bound kinematic shakedown theorem (Symonds and Neal 1951a,b (2) and 2000; Koiter 1960). These provide bounds on a unique value of a load or material parameter that define the shakedown limit. There is, however, a difficulty in the equivalent theory for frictional contact as classical plasticity assumes a convex yield condition and an associated flow rule, i.e. inelastic strain increments are proportional to the gradient of the yield condition. Although Coulomb friction between surfaces defines a convex yield condition, Coulomb slip is not associated. An associated condition applies in only two limiting conditions. When the normal pressure on the slip surface is assumed to be defined by the applied load, the slip condition only involves shear stresses and slip is associated. This case has been discussed by Barber et al. (2008), Klarbring et al.

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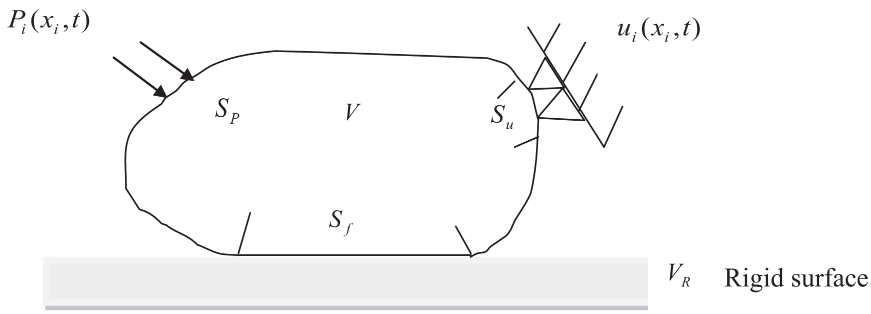

Fig. 1. The general problem.

(2007) and Andersson et al. (2014) and is referred to as the *uncoupled* case. In this case theorems exist analogous to the classical plasticity theorems. The shakedown limit is uniquely defined and the optimal upper and lower bounds coincide. For the coupled case, where both the normal pressure and slip enter into the problem, the slip condition is only associated when a displacement normal to the contacting surfaces is allowed. This is physically unrealistic and contrary to the accepted Coulomb slip conditions. As a result, there is a need for an extension of the classical plasticity theorems that allow a non-associated flow rule, which this paper attempts to provide in the context of this simple problem.

Existing approaches to the *uncoupled case* (Bjorkman and Klarbring (1987), and Flecek et al. (2015)) use the lower bound Melan approach for a history of loading of the form $\lambda P_i(x_i, t) + P_i^0(x_i)$ where the largest value of λ is sort for which a shakedown stress history exists. This is achieved by solving a Linear Programming problem and provides a sufficient condition for shakedown but not a necessary condition. Flecek et al. (2015) have demonstrated, by simulation, that ratchetting may occur at values of λ less than the Melan limit.

The state of residual shear stress at the beginning of a cycle depends in detail on the previous history of loading and initial state. In the classical theorems this initial state is defined, at shakedown, by the residual stresses associated with the optimal lower bound. In the non-associated problem this initial state is not unique and is generally unknown. It is, therefore, necessary to retain this initial state as a variable in the search for possible cyclic solutions. Hence cyclic solutions are not necessarily unique; for a prescribed history of cyclic loading, there may well exist more than one cyclic solution. The primary purpose of this paper is to describe a kinematic theorem, equivalent to the Symond-Koiter theorem, which defines conditions when cyclic slip will definitely not occur, for all possible initial states. On the other hand Melan's theorem (Bjorkman and Klarbring (1987), and Flecek et al. (2015)) defines conditions when shakedown cannot occur, i.e. cylic slip must occur, again independent of initial conditions. These limits do not coincide, and an intermediate condition exists where either cyclic slip or shakedown may occur depending on the initial conditions. This is the behaviour exhibited in the simulations described by Flecek et al. (2015).

If a loading history $\lambda P_i(x_i,t)$ is within shakedown for a particular value of μ, then so is the load history $\alpha\lambda P_i(x_i,t)$ for any (positive) value of α. In other words, shakedown depends on the details of the loading history and not upon its absolute magnitude. Hence for Coulomb friction the appropriate question is as follows. For an entire class of loading histories $\lambda P_i(x_i,t)$ and for any value of λ, what ranges of value of μ and initial distribution of slip $\bar{v}_i$ ensure that shakedown certainly occurs and, alternatively, cyclic slip certainly occurs.

The general structure of the problem is discussed in Section 2 and the properties of the cyclic state are discussed in Section 3. In Section 4, a kinematic work bound in excess of shakedown is derived, i.e. assuming finite slip occurs during the cycle of loading. In Section 5, a kinematic shakedown bound is derived as a limiting case of the general work bound when the slip during the loading cycle becomes infinitesimally small. This bound may be used to define a value of $\mu = \mu^{ks}$ so that shakedown will certainly occur for$\mu > \mu^{ks}$ independent of the initial slip $\bar{v}_i$. In Section 6 the Melan limit$\mu = \mu^{ss}$ is discussed where for $\mu < \mu^{ss}$ cyclic slip will certainly occur independent of the initial slip $\bar{v}_i$. Between these limits$\mu^{ks} > \mu > \mu^{ss}$ either shakedown or ratchetting occurs, depending on the precise distribution of initial slip $\bar{v}_i$. In Section 7 a consistent shakedown limit $\mu = \mu^{cs}$, $\mu^{ks} > \mu^{cs} > \mu^{ss}$, is defined where both kinematic and static conditions for shakedown are simultaneously satisfied. It is possible that more than such limit exists and, in this case the value which give the maximum μ^{cs} becomes the effective shakedown limit.

A summary of the shakedown limits is given in Section 8. The discretised problem for two dimensional problems and a possible approach to constructing consistent cyclic solutions is discussed in Section 9. The proposed method is based upon the Linear Matching Method for the associated shakedown problems. A simple problem is solved in Section 10, demonstrating the main features of the previous theory. To the author's knowledge, this is the first time a shakedown theory has been derived for a non-associated flow rule. The development of the same theory for problems in geomechanics will be discussed elsewhere.

2. The slip problem

Consider the problem shown in Fig. 1. An elastic body with volume V is subjected to external forces or displacement. Over the body's surface S, on part S_T a cyclic history of load is applied, $\lambda P_i(x_j,t)$. On a separate part of S, S_u, a cyclic history of displacement $u_i(x_j,t)$ occurs. Over the remaining flat surface areaS_f, defined by $z = 0$, frictional condition occurs against an adjacent rigid body V_R. The tangential frictional traction obeys Coulombs law and is related to the direction of tangential movement $v_i = (v_x,v_y)$ of the elastic body relative to the rigid surface. The compressive inter-surface pressure $p = -\sigma_{zz}$ and the surface shear stresses are denoted by$q_i = (q_x,q_y) = (\sigma_{zx},\sigma_{zy})$ as shown in Fig. 2. It is assumed that the pressure between the two surfaces is compressive, $p \geq 0$ and the normal displacement $v_z = 0$. Slip conditions are given by:

$$\text{Slip} \quad |q_i| = q^s = \mu p \ \text{ and } \ \dot{v}_i = q_i\left(\frac{|\dot{v}_i|}{q_s}\right), \tag{2.1}$$

$$\text{Stick} \quad |q_i| \leq q^s \ \text{ and } \ \dot{v}_i = 0, \tag{2.2}$$