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Design maps for fracture resistant functionally graded materials

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ABSTRACT

The objective of this research is to generate design maps to identify functionally graded microstructures with enhanced fracture toughness. Several Functionally Graded Materials (FGMs) with an edge crack and membrane loading are considered and the resulting *J*-integral values are computed numerically using Finite Element Analysis. In order to capture the resulting stress fields accurately, Barsoum elements are used in the vicinity of the crack tip and the simulations are carried out for several crack lengths (a) and material contrasts (κ). The averages of the *J*-integral values are used to determine the normalized Stress Intensity Factors which are then benchmarked with existing analytical solutions in special cases. Furthermore, in order to facilitate an objective comparison between FGMs and homogeneous materials, a constraint is imposed on each of the microstructure so that the volume averaged modulus remains the same although the spatial variation is very different. Subsequently, we demonstrate that a FGM could perform either better or worse than the reference homogeneous material depending upon the crack length, the type of functional gradation and the material contrast (thereby the local gradient of the modulus at the crack tip). Finally, the notion of 'Fracture Index' is introduced using which 'design maps' are created in the (a–κ) space that reveal microstructures with enhanced fracture resistance. These maps are universal since any Functionally Graded Material can be mapped as a point on this space.

1. Introduction

Functionally Graded Materials are a subset of inhomogeneous materials and have unique mechanical, thermal and electrical properties due to the controlled spatial variation of material properties. This is in contrast to several homogeneous materials that typically have lower structural integrity when compared to their FGM counterparts. In fact, inhomogeneous materials are ubiquitous in nature and most materials exhibit some form of inhomogeneity when considered at appropriate length scales. Functionally Graded Materials have been found to be useful in several applications in engineering and medicine. For instance, FGMs have been used for synthesizing thermal barrier coating for space applications (Luo et al., 2014; Krenkel and Berndt, 2005), design of columns (Ranganathan et al., 2015; Aldadah et al., 2014), piezoelectric and thermoelectric devices (Bodaghi et al., 2014; Gasik and Bilotsky, 2014) and in dental implants which have demonstrated superior mechanical behavior, biocompatibility and osseointegration improvement (Mehrali et al., 2013; Watari et al., 2004). Also, with the advent of additive manufacturing, functionally graded bone implants with superior fracture resistance can be printed in a cost-effective manner (Sing et al., 2015; Bartolo et al., 2012). It is therefore imperative to understand the behavior of FGMs in order to effectively design composites for target applications.

In the past, several techniques have been used to examine the fracture of FGMs and some of these are noteworthy. Gibson (1967) laid the foundation for analyzing FGMs by modeling soil as a heterogeneous material. In the study, the author examines an elastic half space in which the Young's Modulus, E is varied with depth. Analytical and semi-analytical approaches were subsequently used by Delale and Erdogan (1983) and Erdogan and Wu (1997) in order to investigate cracks in heterogeneous materials. In particular, Delale and Erdogan (1983) derive the integral equation for mode-I loading and prove that the Poisson's ratio, ν has a negligible impact on the Stress Intensity Factor (SIF). By exponentially varying the Young's Modulus in the direction parallel to the crack, the authors demonstrate that the crack surface displacement is lower in the stiffer portion of the heterogeneous material as compared to the homogeneous material and vice-versa. Along similar lines, Erdogan and Wu (1997) analyze a FGM that has a crack perpendicular to the

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boundaries. The Young's Modulus is varied along the thickness of the material in order to obtain the results under various loadings such as fixed grip, membrane and bending. The mode-I SIF is obtained for edge cracks and it is seen that the inhomogeneity in the material directly impacts the stress distribution and the SIF. For membrane and bending loadings, the SIF is higher for lower material contrasts and is lower for higher material contrasts. This is in contrast to fixed grip loadings where the trend is vice-versa.

An alternate approach to analyze fracture using the Element Free Galerkin's Method (EFGM) was proposed by Rao and Rahman (2003). In their work, a novel approach was used to obtain the SIF for a 2D stationary crack in a heterogeneous medium. This involved spatially varying the material parameters as smooth functions and obtaining several integral equations for analyzing mixed mode fracture. Five problems were taken into consideration and the results obtained for mode-I and mixed mode fracture were compared with analytical as well as numerical solutions. It was seen that EFGM results were in good agreement with the reference solutions.

Several authors have also employed numerical simulations based on Finite Element Analysis to model fracture in FGMs. In particular, Kim and Paulino (2002) examined stationary cracks in a FGM and varied the elastic moduli as a smooth function of spatial coordinates. This was then incorporated into the stiffness matrix and the SIF for mode-I along with mixed mode cases were compared using three new approaches for analyzing fracture: i) path independent *J*-integral method for inhomogeneous materials; ii) modified crack-closure integral method (MCC); iii) displacement correlation technique (DCT). The authors concluded that for mode-I case, the solution obtained using any of these three approaches were identical.

Similarly, Anlas et al. (2000) performed Finite Element Analysis and obtained the *J*-integral values for a FGM plate with an edge crack in order to calculate SIF. Numerical simulations were carried out using ABAQUS and the domain was discretized into forty parts to model the gradation in the Young's Modulus. Elements consisting of four nodes and four integration points were used for obtaining the results. Further, the authors discussed that the *J*-integral was path dependent for an inhomogeneous material. However, the value for the *J*-integral could be used to determine the SIF for a heterogeneous material in the vicinity of the crack tip. Finally, the authors benchmarked their results for uniform traction and uniform displacement loadings and were successful in obtaining accurate results.

More recently, Hossain et al. (2014) proposed a new methodology to determine the toughness of a heterogeneous medium that was independent of the details of the boundary condition. Instead, the authors numerically simulated a domain by applying a surfing boundary condition to a steadily growing macroscopic crack. Subsequently, the effective toughness was obtained by using the energy release rate required to propagate the crack.

In summary, a significant amount of literature exists for determining the fracture toughness of FGMs. However, to the best of our knowledge there is no framework to unify the treatment of FGMs in order to identify microstructures that maximize fracture toughness. The notion of ‘Fracture Index’ and ‘design maps’ will be introduced in this paper to fill this void in existing literature. In the subsequent sections, we develop the necessary mathematical models and the resulting boundary value problems are then solved numerically using Finite Element Analysis. The results obtained are first benchmarked with existing solutions for special cases. After validating the numerical model, the ‘Fracture Index’ is computed for Functionally Graded Materials as a function of crack length and material contrast. This procedure is repeated for a variety of functional distributions to generate the ‘design maps’. It will be demonstrated that these maps highlight the admissible regions for designing microstructures with enhanced fracture resistance.

2. Mathematical background

The *J*-integral is widely used in order to accurately obtain Stress Intensity Factors (SIFs) at crack tips (see Abd-Elhady and Sallam (2015), Aliha et al. (2013) and Dancette et al. (2012)). It is an alternative to the strain energy release rate and was first proposed by Rice (1968) in order to analyze cracks. For Functionally Graded Materials, Delale and Erdogan (1983) studied the problem of Mode-I loading with a variation of Young's Modulus in the direction parallel to the crack. The authors stated “it is reasonable to expect that in nonhomogeneous materials with continuous and continuously differentiable elastic constants the nature of the stress singularity at a crack tip would be identical to that of a homogeneous solid”. Similarly, Erdogan (1983) stated that if a crack was embedded into an inhomogeneous medium with smoothly varying elastic properties, the square root nature of the stress singularity seemed to remain unchanged.

The conjecture by Delale and Erdogan was analytically proven by Eischen (1987) using an eigenfunction expansion technique similar to that of Williams (1957). In the study, Eischen considered a constant Poisson's ratio and a general functional form of the Young's Modulus variation. The author proved that a $r^{-1/2}$ stress and strain singularity existed at the crack tip, r being the radial distance measured from the crack tip. In addition, the angular variation of the singular stress field and the associated displacements around a crack tip in a Functionally Graded Material were shown to be exactly the same as the angular variation in a homogeneous material (see Eischen (1987) and Honein and Herrmann (1997)). With this framework laid out, we will now demonstrate the *J*-integral in two dimensions and its relation to the Stress Intensity Factor. In general, the *J*-integral can be defined as (Shih et al., 1986; Abaqus Theory Manual J-I, 2012)

$$J = \lim_{\Gamma \to 0} \int_{\Gamma} \mathbf{n} \cdot \mathbf{H} \cdot \mathbf{q} \, d\Gamma \tag{1a}$$

where

$$\mathbf{H} = W\mathbf{I} - \boldsymbol{\sigma} \cdot \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \tag{1b}$$

and Γ is a contour around the crack tip (see Fig. 1), the limit $\Gamma \to 0$ indicates that Γ shrinks onto the crack tip, $\mathbf{q}$ is a unit vector in the virtual crack extension direction, $\mathbf{n}$ is the outward normal to Γ, $\mathbf{I}$ is the Kronecker Delta, W is the strain energy density, $\boldsymbol{\sigma}$ is the stress

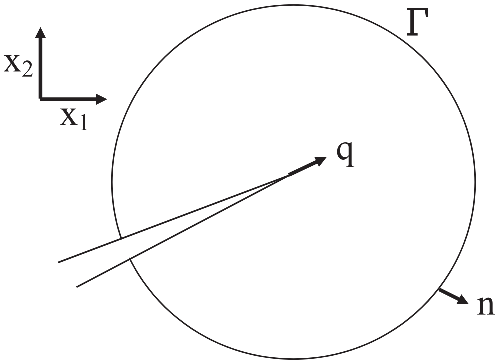

Fig. 1. Contour for evaluation of the *J*-integral.