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Convergence of viscoelastic constraints to nonholonomic idealization

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ABSTRACT

Rolling motion, which is usually described by means of nonholonomic constraints, can occur in many technical systems e.g. roller bearings or gear wheels in gearboxes. The idealized modeling of mechanical multibody systems with rolling elements leads to differential algebraic equations (DAEs). The kinematical condition of a vanishing relative velocity is enforced by constraint forces. However contact areas are not ideally rigid, but compliant due to local deformations of asperities and elasticity of the contacting bodies. For this reason a sensible physical description should take these effects into account. Thus the contact forces are modeled in the present paper using a tangential viscoelastic force element in the contact. The rolling motion is then enforced by applied forces, instead of constraint forces in the ideally rigid case. The objective of this work is to show that under certain conditions, solutions of the general multibody system with viscoelastic contact model converge to the solutions of the multibody system containing idealized nonholonomic constraint equations, if the viscoelastic constants approach infinity. An ansatz in form of an asymptotic series expansion with initial layer terms is introduced to prove the convergence under appropriate assumptions on the viscoelastic parameters. In order to suppress high frequency oscillations in the contact, the choice of the damping parameter is inspired by the critical damping, known in linear systems theory.

1. Introduction

Rolling contacts are usual in various technical systems. Gear wheels in gearboxes, the motion of rolling elements in roller bearings or the Euler disk (Caps et al., 2004) or even car tyres can be mentioned here as examples. In most cases the constraint equations on velocity level enforcing a rolling motion cannot be integrated, yielding nonholonomic constraint equations. Usually the nonholonomic constraints can be incorporated into the equations of motion by the method of Lagrange multipliers. This leads to the equations of motion for a constrained multibody system with m nonholonomic constraint equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} - Q + G(q)^T\lambda = 0,$$
$$G(q)\dot{q} = 0.$$

Thus, the problem at hand is an index-2 differential algebraic problem. The investigation of nonholonomic mechanical systems led to a large variety of subspace methods. The main target of these methods consists in introducing quasi coordinates or generalized coordinates or velocities in order to obtain a system of ordinary differential equations instead of differential algebraic equations. If there are n generalized coordinates and m non-integrable constraints, the changes of momentum and moment of momentum evolve on an $n-m$ manifold and are mapped into the n dimensional coordinate space via kinematic equations. A path-breaking work concerning nonholonomic mechanical systems was the work done Caplygin (1897), see also Neimark and Fufaev (1967). His interest was devoted to a special class of conservative nonholonomic systems, described by n generalized coordinates, where only the first m generalized velocities can be regarded as independent. The equations of motion were derived under the following assumptions: The Lagrange function and the constraint matrix do not depend on the integrals of the dependent velocities. Those systems are usually referred to as Caplygin systems. Under certain circumstances it may be useful to introduce a set of quasi coordinates instead of generalized coordinates in order to reduce the size of the equations of motion. A well known example are for instance Eulers equations, where the angular velocities are introduced as quasi coordinates instead of a set of

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angles. This was already done by Caplygin. There are further approaches based on quasi coordinates, like the method introduced by Hamel (1903). Voronets (1901) developed an approach to derive the equations of motion of a mechanical system with nonholonmic constraint equations without the limiting assumptions, that were made by Caplygin. Appell (1900) introduced a more general approach for the derivation of the equations of motion of a mechanical system in quasi coordinates, that holds for both holonomic and nonholonomic systems. The approach leads to a set of equations, that are usually referred to as Appell-Gibbs-Equations. These equations were first introduced by Gibbs (1879), but only for holonomic systems. Maggi (1901) proposed a method to derive the equations of motion for a mechanical system with nonholonomic constraints. The last approach, that shall be mentioned here are Kanes dynamical equations, that can be regarded as a special case of Maggis equations, see Kurdila et al. (1990). Kane and Levinson (1985) derived them for both holonomic and nonholonomic systems. For a Lagrangian derivation of Kanes equations see Parsa (2007). All these subspace methods describe the exact dynamics of the underlying system in the rigid sense, without taking contact compliance, especially in tangential direction, into account.

Regarded from a physical point of view, nonholonomic motions are mainly caused by friction forces. These can be modeled in various ways. Usually viscous damping forces are used in order to describe a sticking state. However it may be useful from a physical point of view, to take tangential compliance of a contact into account. Caratheodory (1933) came to the conclusion, that it is impossible to approximate the motion of a sleigh, constrained by a nonholonomic constraint, by modeling the friction force as viscous damping. However Caratheodory's arguments cannot be regarded as convincing, as is shown by Neimark and Fufaev (1967). Later Karapetian (1981) showed, that Tikhonov's theorem can be applied in order to prove the convergence of the viscous solution to the idealized DAE solution, in case of describing the friction forces by means of pure anisotropic viscous damping, if the dissipation coefficient approaches infinity. Kozlov (1983) investigated various ways of passages to the limit. There are nearly an uncountable number of publications concerning the convergency of the solution of viscous friction models to the solution of the corresponding differential algebraic equation. Levin and Levinson (1954) investigate a system where the parameter $\varepsilon > 0$ multiplying the highest derivative is of power $r > 0$. Convergency of the regularized problem to the differential algebraic problem is proven. For $r = 1$ Hoppenstaedt (1966) obtains quite similar results. O'Malley and Flaherty (1980) use a series expansion with initial layer terms to prove convergency. There exist also various approaches and publications concerning penalty methods for differential algebraic problems. Baumgarte (1972) introduced a method to stabilize the constraints. This method can be applied to holonomic and nonholonomic constraint equations as well. Baumgartes idea is based on an index reduction method. Ostermeyer (1983) proposed approaches based on control theory that regulates the constraint error to zero. Lötstedt (1985) proposed a method, that can be regarded as a combination of Baumgartes method and singular perturbations. He is capable to prove the convergency of the regularized solution. Another approach to regularize a general index 3 differential algebraic problem in Hessenberg form is proposed in Knorrenschild (1988). The main target of his work consists in finding a derivative-free index reduction method. He introduces a parasitic perturbation in the constraint equation which results in a differential algebraic equation of index 2. Repeated introduction of the parasitic perturbation in the equation finally results in an index 1 problem, without having differentiated any equation. Hanke and Eich (1994) used a quite similar approach. All of these results were obtained for the case where the nonholonomic constraint is approximated by tangential viscous damping forces.

However it is important to mention here, that our main target does not consist in avoiding the need to solve a differential algebraic problem. Instead our main target is to take both tangential viscous damping forces and tangential contact elasticity into account. In the present paper we investigate a viscoelastic idealization of nonholonomic constraints, that is motivated by physical considerations. Two solids with convex surfaces rolling on each other possess a tangential contact compliance, which is in contrast to the usual penalty methods taken into account. In our case the viscoelastic tangential contact model is not attributed to one of the solids, instead it is modeled as an averaged contact compliance. Pure rolling is equal to sticking, with kinematically repositioned contact point. In case of pure viscous damping, the contact point is always in the sliding state, which contradicts real physical behavior. Usually sticking is modeled by introducing an elasticity in the contact as demonstrated by Vielsack (1996). Here the constraint is mainly enforced by the elasticity, the dissipative terms help to avoid oscillations in the contact. In an earlier work Stamm (2011) applied this kind of viscoelastic formulation to a tangential contact law, extending the classical laws of friction, like the Coulomb model, to distributed contacts, in order to circumvent the problem of indeterminacy in the sticking state. It would be advantageous, if the idealized rigid formulation approximated the description of a contact law by means of viscoelastic forces in case of infinitely stiff chosen viscoelastic parameters. Thus the objective of this work is to show the convergence of the viscoelastic description to the idealized nonholonomic rigid description in a strictly mathematical sense. The paper is structured in the following way. In Section 2 the statement of the problem is given. After the derivation of the viscoelastic description, a theorem is proven, which states convergency of the viscoelastic description to the ideally rigid description. In Section 3 the statement of the proof is confirmed by two numerical examples. In case of the first example, an analytical solution exists, which is compared to the numerical results.

2. Statement of the problem

The motion of a general multibody system underlying linear nonholonomic constraint equations can be described by means of the following index-2 differential algebraic equation system:

$$M(\hat{q})\ddot{\hat{q}} = F(\hat{q}, \dot{\hat{q}}, t) - G^T(\hat{q})\Lambda, \quad t \in I = [t_0, t_e], \tag{1a}$$

$$0 = G(\hat{q})\dot{\hat{q}} \tag{1b}$$

$$\hat{q}(t_0) = \hat{q}_0, \quad \dot{\hat{q}}(t_0) = \dot{\hat{q}}_0. \tag{1c}$$

where $M(q) \in \mathbb{R}^{n\times n}$ denotes the symmetric and positive definite mass matrix, $G(q) \in \mathbb{R}^{m\times n}$ the constraint matrix and $F(q, \dot{q}, t) \in \mathbb{R}^n$ contains all external forces acting on the system. Here and in the following use the short notation

$$\dot{\varphi}(\psi) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(\psi(t))$$

for differentiable functions φ of and ψ. Moreover, we omit the arguments of M, F, and G if they are clear from the context.

Assumption 1 (Differential algebraic initial value problem). *We assume that $G(q) \in \mathbb{R}^{m\times n}$ and $F(q, \dot{q}, t) \in \mathbb{R}^n$ are sufficiently smooth, the matrix $G(q)$ is assumed to have full rank m. Moreover, we assume that the initial data* (1c) *is consistently chosen.*