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# A structural model for plane sandwich beams including transverse core deformability and arbitrary boundary conditions

Ornella Mattei, Lorenzo Bardella*

*Department of Civil, Environmental, Architectural Engineering and Mathematics (DICATAM), University of Brescia, Via Branze, 43, 25123 Brescia, Italy*

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ABSTRACT

In order to model the effect of arbitrary boundary conditions on plane linear elastic sandwich beams, we develop a structural theory relying on a zigzag warping: each layer, of arbitrary thickness and modulus, is described by the Timoshenko kinematics and, for the core, we further consider the transverse strain, which measures the normal deformability along the core thickness. This structural model, dependent on six functions of the beam axis coordinate, builds upon the theory put forward by Dusan Krajcinovic in the early Seventies. By following a variational approach, we obtain and discuss the (Euler–Lagrange) balance equations and the (natural) boundary conditions governing the model. In sandwich beams having a soft core, this model can describe relevant features of the stress state due to "severe boundary conditions", including, for instance, loading on a specific skin coupled with constraints realised, at certain cross-sections, on the opposite skin only. In this work we focus on the flexure accompanied with non-uniform shear. In particular, we consider the cases of cantilever and propped-cantilever beams subject to uniform load. We provide accurate shear stress estimates by post-processing, through a Jourawski-like approach, the longitudinal normal stress predicted by the beam model. We demonstrate the capability of the proposed model by comparison of its results, obtained by using the Rayleigh–Ritz method, with those of continuum plane stress Finite Element (FE) simulations. The predictions of the present beam model are shown to be useful at fully clamped cross-sections, where displacement-based FE results are unreliable.

## 1. Introduction

We aim at developing a structural model able to describe the linear elastic response of any plane sandwich beam, with perfect interfaces, subject to bending and shear. In particular, since each structural model available in literature is reliable only for a limited range of relative stiffness between core and skins (Tonelli et al., 2012), in this investigation we seek to lay the basis for developing a model to accurately estimate the stress field for every choice of the material and geometrical properties, granted that the skins are stiffer than the core. This is important if the stress field has large spatial gradients ensuing from specific boundary conditions. Among these, for instance, it is of particular interest the case of a support condition realised on the skin opposite to that the load is applied on. This is a typical example of what here we refer to as *"severe boundary conditions"*, that is a distinctive feature of the present investigation.

In order to illustrate the proposed structural model, let us now introduce some basic assumptions, along with some notation. The longitudinal beam axis, say $x$, is chosen to coincide with the core centre-line, while $y$ is the coordinate along the cross-section height. The top and bottom layers, called the skins or faces, may be geometrically and mechanically different and they may be *thick* (Allen, 1969); we denote their respective thicknesses as $t_\mathrm{u}$ and $t_\mathrm{l}$, and their respective longitudinal Young's moduli as $E_\mathrm{u}$ and $E_\mathrm{l}$. Here and henceforth, the subscripts u and l are used to refer to the upper (or top) and the lower (or bottom) layers, respectively. The core is assumed to be isotropic, with Young's modulus $E_\mathrm{c}$ and Poisson's ratio $\nu_\mathrm{c}$. We consider a plane stress mechanical response, in the $(x,y)$ plane, of sandwich beams subject to a static transverse load per unit length $q(x)$, acting along $y$. Beside the shear and longitudinal normal strains and stresses within each layer, the model here presented accounts for the normal strain, $\varepsilon_y$, and stress, $\sigma_y$, in the core. This allows the description of sandwich beams whose core is

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* Corresponding author. Tel.: +39 0303711238; fax: +39 0303711312.
*E-mail address:* lorenzo.bardella@ing.unibs.it (L. Bardella).

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much softer than the skins, as in the case of cores made of foam. The fundamental assumptions of the adopted kinematics are displayed in Fig. 1, and rely on a zigzag warping. Such kinematics can be described by the following six independent functions:

$$u_{\mathrm{l}}(x), \quad u_{\mathrm{u}}(x), \quad v(x), \quad \phi_{\mathrm{l}}(x), \quad \phi_{\mathrm{u}}(x), \quad s(x) \tag{1}$$

in which $u_{\mathrm{l}}(x)$ and $u_{\mathrm{u}}(x)$ are the axial displacements of the centre-lines of the lower and upper layers, $v(x)$ is the deflection of the sandwich longitudinal axis $x$, $\phi_{\mathrm{l}}(x)$ and $\phi_{\mathrm{u}}(x)$ are the rotations of the lower and upper layers, and $s(x)$ is the structural variable describing the core deformability $\varepsilon_y$, henceforth called the transverse core strain. In this work, as a first step toward the abovementioned aim, we assume the transverse core strain to be independent of $y$, that is $\varepsilon_y \equiv s(x)$, with the purpose of establishing what such a “simple” modelling can predict. Under this assumption, $s(x)$ is the jump between the transverse displacements of the bottom and top layers divided by the core thickness $c$.

The kinematics described above is richer than that formerly proposed by Yu (1959) and then developed by Krajcinovic (1972, 1975). They both considered sandwich beams with identical skins and neglected $\varepsilon_y$, so that the rotations of the top and bottom layers are equal and the structural model turns out to be described by three kinematic variables only. Actually, Krajcinovic (1972) put forward the possibility of accounting for $\varepsilon_y$ in the core within the zigzag structural theory but, to the best of our knowledge, such a specific theory has so far remained undeveloped in literature. Here, we investigate on such extension of the Krajcinovic (1972) model. It is worth noting that the novelty of the present work does not lie in the asymmetry, with respect to $z$, of the sandwich cross-section, as the extension of the Yu–Krajcinovic model to the case of unequal skins is conceptually simple (albeit computationally entangled due to the introduction of further unknown functions), while it lies in accounting for the transverse core strain within a structural theory, applied to boundary value problems characterised by “severe boundary conditions” (and fully clamped cross-sections, deserving particular attention, as discussed later).

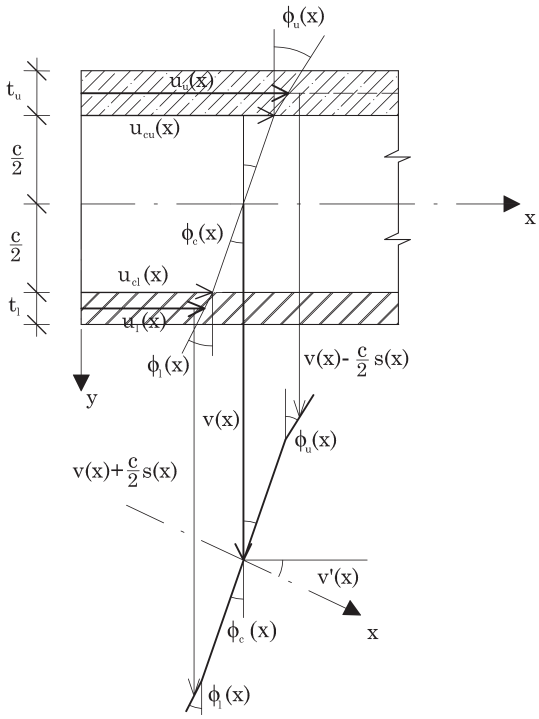

**Fig. 1.** Zigzag kinematics for a sandwich beam with unequal skins and core deformable along the y direction.

For more than 20 years Frostig and co-workers have been developing sandwich models accounting for the transverse core deformability. In the pioneering work of Frostig et al. (1992), contrary to the model presented here, the core is modelled as a two-dimensional plane stress continuum (so that the transverse core strain depends both on $x$ and $y$), the skins are supposed to be *thin* (Allen, 1969), so that they are modelled as Euler–Bernoulli beams, and the longitudinal normal stress $\sigma_x$ in the core is neglected, the sandwich being assumed to be “antiplane” (Allen, 1969; Serpilli and Lenci, 2008; Berdichevsky, 2010; Tonelli et al., 2012). This last assumption implies that the shear stress $\tau_{xy}$ in the core is independent of $y$, i.e., uniform along the thickness. Instead, the model presented in this work accounts for $\sigma_x$ in the core, that is, we consider also non-antiplane sandwiches. The model developed here is, in fact, more similar to that of Phan et al. (2012), proposed in one of the most recent investigation of Frostig and co-workers. The beam model of Phan et al. (2012) depends on seven functions of the beam axis and has the peculiarity of employing, for the core kinematics, terms up to the cubic power of $y$ for the axial displacement component and up to the second power of $y$ for the transverse displacement component, that is significantly richer (and computationally more expensive) than the core kinematics adopted here. Instead, Phan et al. (2012) retain the thin skins assumption, while here we account for possibly thick faces, modelling them as Timoshenko beams. Furthermore, Phan et al. (2012) restrict the attention to the case of a simply-supported beam subject to sinusoidal transverse load, with the support condition uniformly applied to all the layers (“smooth boundary conditions”), looking for analytic solutions.

Very recently, Jedari Salami et al. (2016) developed a much richer sandwich beam model, in which the kinematics adopted by Phan et al. (2012) for the core is assigned to any layer, thus employing twenty-one structural functions. However, such structural functions are constrained in such a way as to *a priori* satisfy the equilibrium conditions at the interfaces and at the upper and lower sandwich sides. Jedari Salami et al. (2016) do not discuss the Euler–Lagrange equations ensuing from their model as they focus on numerically minimising the Total Potential Energy governing their model to obtain a solution for the three-point bending case, considering “smooth boundary conditions”. In the comparison with a Finite Element (FE) simulation adopting continuum elements, Jedari Salami et al. (2016) show a large discrepancy in terms of shear stress near the support.

The relevance of the transverse core strain in multilayer beams has been also analysed in the context of *equivalent single layer theories* by Vidal and Polit (2009), which, contrary to the *layerwise* approach, aim at modelling laminated structures by keeping the number of independent unknown functions unrelated to the number of layers (see, e.g., Zuo and Hjelmstad, 1998; Ghugal and Shimpi, 2001; Tessler et al., 2009; Carrera et al., 2013).

In principle, the layerwise-zigzag approach here followed suffers a drawback: the structural model alone may provide poor estimates of the shear stress. In fact, although such zigzag theories are widely used in laminated beams for being extremely accurate in describing both the deflection and the normal longitudinal stress (see, e.g., Heller, 1969, Sharma and Rao, 1982; Bardella and Tonelli, 2012; Kristensen et al., 2008; Galuppi and Royer-Carfagni, 2012), they predict uniform shear stress in the skins (and also in the core if $\varepsilon_y$ is neglected), thus violating the boundary and interface equilibrium conditions. Nevertheless, such a shortcoming can be overcome as for instance recently proposed by Bardella and Tonelli