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Full weak uniqueness in anisotropic time-dependent Mindlin theory for plates

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ABSTRACT

The equations of a dynamic Mindlin theory for the bending of anisotropic plates are presented. The elastic coefficients are assumed to satisfy 3D triclinic symmetry conditions plus additional assumptions in order to deduce the 2D constitutive equations for the plate. The uniqueness is proved in a full weak form, relative to both space and time co-ordinates, without any assumption of positive-definiteness, on the basis of the symmetry relations satisfied by the elasticity tensor.

1. Introduction

The different aspects of Mindlin theory of plates has been widely investigated in many papers since decades. Uniqueness results in the theory of plates are established in various conditions; with or without positive-definiteness assumption, weak or strong solutions, different cases of anisotropy (monoclinic, orthotropic, isotropic), Kirchhoff (for thin) or Reissner-Mindlin (for thick) plates, linear or even non-linear theory (for some particular loads and shapes of the plate) (Matkowsky and Putnick, 1975; Eringen and Şuhubi, 1975; Constanda, 1986, 1987; Schiavone, 1991; Bielski and Telega, 1996; Ciarlet, 1997; Ebenfeld, 1999; Ciarletta, 1999; Chudinovich and Constanda, 2000; Fu, 2003; Wu, 2004; Mindlin, 2006).

In the above cited papers, in order to prove the uniqueness, the assumption of a positive definite elasticity tensor is generally fulfilled, with the exception of Ciarletta (1999), where the symmetry of the elasticity tensor is used, and Matkowsky and Putnick (1975), based on the energy integral method. The mentioned assumption allows to prove uniqueness not only for strong solutions but for weak solutions also, together with the existence, through powerful tools as Riesz representation theorem or Lax-Milgram theorem (when the symmetry is missing, Evans (2010), p. 317).

Concerning specifically Mindlin plates, we mention the interesting uniqueness results for strong solutions, established without the positive definiteness assumption in Passarella and Zampoli (2009b) for transversely isotropic plates and Passarella et al. (2010) for rhombic and strongly elliptic plates. The first result is proved for a bounded domain of the plate, while the second one works for both bounded and unbounded domains.

Certain conditions must be satisfied so as a plate theory to be set up and to work accurately enough. An example of analysis of some conditions involving the applicability of Reissner-Mindlin and Kirchhoff-Love bending theories of plates can be found in Arnold et al. (2002).

In the 3D elastodynamics, a higher variety of results are obtained through different methods, in elasticity, thermoelasticity, viscoelasticity. If the goal is uniqueness without positive-definiteness assumption, the Lagrange identity method and the logarithmic convexity method are very useful. Uniqueness for strong solutions was obtained with the first method (Brun, 1969; Rionero and Chiriţă, 1987), and uniqueness for strong and weak solutions resulted with the second one in Levine (1970), respectively Knops and Payne (1968).

Using the positive-definiteness assumption, theorems of existence and uniqueness were established in Eremeyev and Lebedev (2013) and Altenbach et al. (2010) for weak solutions defined in proper energy spaces. The results of these authors hold for equilibrium, vibration or dynamic problems and for different boundary conditions of special significance like (partially) clamped, free or supported elastic bodies with or without surface stresses.

There are more examples of papers where the Lagrange identity method is used as a main tool. In Knops (2001), uniqueness is

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established for strong solutions, supposing only a partial positive definiteness. In Levine (1977), a uniqueness result is proved for weak solutions in some Hilbert spaces, without positive definiteness or energy conservation law. Uniqueness for strong solutions in unbounded domains is obtained in Galdi et al. (1986). Analysing nonsimple thermoelastic materials, Marin et al. (2013) established the uniqueness also in a strong frame. The Lagrange identity method is used in Knops (1988), dealing with estimates for continuous data dependence. In Payne (2006), various mathematical tools, including the mentioned method, are carefully analysed in the frame of some improperly posed problems. As a general conclusion from these papers, this method allows to avoid assumptions like positive definiteness and energy conservation laws. Lagrange identity method is applied in the present work as well.

A true collection of classical uniqueness results is found in Knops and Payne (1971), for both elastostatics and elastodynamics in various conditions as those already mentioned and more. The weak uniqueness from Knops and Payne (1968) is also recalled.

In the present paper, for an elastodynamic Mindlin plate theory of bending, we obtain a uniqueness result in a full weak form, relative to both space and time co-ordinates. The solution is searched in an appropriate Sobolev subspace, the boundary and the initial conditions of the dynamic problem being defined in the sense of trace. In the proof, we use only the symmetry of the elastic tensor, without positive-definiteness or some conservation law for the energy. Also, we considered bounded and unbounded time intervals.

Among the above-cited papers, only Knops and Payne (1968) and Levine (1977) deal with uniqueness for weak solutions, avoiding in the same time any assumption of positive-definiteness. However, their solutions are weak relative to the space co-ordinates only, being assumed strongly differentiable relative to time.

In the next section, we present the equations of an anisotropic Mindlin theory. The main result, namely the uniqueness for weak solutions, is developed in Section 3. Some discussions and conclusions are reserved for the Section 4.

2. Notations and equations

We consider a plate of thickness h, for which the middle plane contains the axes OX and OY of the Cartesian frame, the axis OZ being perpendicular on this plane, its sense being downwards. Therefore, the plate is identified with the body volume $V=\overline{\Omega}\times[-h/2,h/2]$, where $\Omega\subset\mathbb{R}^2$ is a domain (open, bounded, connected subset with a Lipschitz-continuous boundary $\partial\Omega$, Ciarlet, 1988). The smoothness of the boundary has to ensure: s1) a unit outer normal vector field exists almost everywhere (a.e.) along $\partial\Omega$. s2) the divergence theorem (Green formula) can be applied.

If the set Ω is a domain, then s1) and s2) are fulfilled. Assumption s1) is necessary to define the normal and the tangent components along the boundary of different quantities like rotations, shear force and bending and twisting moments (see formulas (9), (10), (11) in this section). The boundary conditions (12) depend on these components. Green formula allows turning a boundary integral into an integral on the domain of the plate, in order to prove that the strong solution defined in this section is weak too (as defined in the Section 3). The same Green formula is used in the subsection ii) of Section 3, in the proof of our uniqueness theorem. The Bernard theorem (2011), which we used in Section 3, requires Lipschitz-continuous boundary for the domain.

For the points of the middle plane we have $(x,y)\in\overline{\Omega}$ and $z=0$. The unit vectors along the three axes OX, OY and OZ are denoted **i**, **j**, respectively **k**. The plate is supposed non-homogeneous, the density satisfying $\rho=\rho(x,y,z)>0,\ \forall(x,y,z)\in V$.

In this section, all the functions are considered differentiable enough so that the calculations have sense. Thus, we take $C_{ijkl}=C_{ijkl}(x,y,z)\in C^0(V)\cap C^1(\Omega),\ \ \rho\in C^0(V)$.

By convention, repeated indices mean summation. The Latin indices take the values 1,2,3 and the Greek ones take only the values 1,2. In the whole paper, the dot above some quantities means pointwise (strong) derivative with respect to time. In this section, a comma in an expression like $f_{,\alpha}$ indicates pointwise derivative with respect to co-ordinates, namely $\partial f/\partial x$ for $\alpha=1$ and $\partial f/\partial y$ for $\alpha=2$.

Customary notations in the theory of elasticity and theory of plates are used throughout the paper (Ciarlet, 1997; Adams and Fournier, 2003). For simplicity, when confusions are not possible, the dependence of the functions on their arguments will not be written.

Since is about the Mindlin theory of bending, we take the displacements of the form:

$$u_1=z\psi_1(x,y,t),\quad u_2=z\psi_2(x,y,t),\qquad u_3=u_3(x,y,t),\quad (x,y,t)\in\overline{\Omega}\times[0,T), \tag{1}$$

where $0<T\leq\infty$ and ψ_1, ψ_2 are rotations around the axes OY and $-OX$, respectively. The unknown functions are taken in the set: $\psi_\alpha,u_3\in C^2(\Omega)\cap C^1(\overline{\Omega})\cap C^2([0,T))$.

The equations of motion for the bending of plates (Yu, 1996; Passarella and Zampoli, 2009b), including the rotatory inertia (the right terms containing $\ddot{\psi}_\beta$) are:

$$M_{\alpha\beta,\alpha}-Q_\beta=\frac{\overline{\rho}h^3}{12}\ddot{\psi}_\beta,\ \ \beta=1,2,\quad \text{in}\,\Omega\times[0,T),Q_{\alpha,\alpha}+q=\rho_3h\ddot{u}_3, \tag{2}$$

$M_{\alpha\beta}$ being the bending and twisting moments, Q_α the shear forces defined below in Eq. (7) and q the transversal load on the plate, $q\in C^0(\overline{\Omega}\times[0,T))$. In the Eq. (2), the body force along the axis OZ (normal to the plate) is integrated in the load q (Passarella and Zampoli, 2009a), while the contribution of the body forces along the axis OX and OY (in the middle plane of the plate) is neglected. Also

$$\frac{\overline{\rho}h^3}{12}\overset{d}{=}\int_{-h/2}^{h/2}\rho z^2\mathrm{d}z,\quad \rho_3h\overset{d}{=}\int_{-h/2}^{h/2}\rho\mathrm{d}z. \tag{3}$$

As have been defined, $\overline{\rho}$ and ρ_3 are opportune average densities. We denote with ε_{ij} and σ_{ij}, the components $\varepsilon_{ij}=(u_{i,j}+u_{j,i})/2$ of the linearized (infinitesimal) strain tensor and of the stress tensor in linearized 3D elasticity, respectively.

In order to deduce the constitutive equations for the plate, some assumptions are made:

- e1) $C_{3333}\neq0$ in V.
- e2) The symmetry conditions for triclinic (general anisotropic) materials are satisfied: $C_{ijhk}=C_{hkij}$, $C_{ijhk}=C_{jihk}$, $C_{ijhk}=C_{ijkh}$.
- e3) In the 3D constitutive equations $\sigma_{ij}=C_{ijhk}\varepsilon_{hk}$ (generalized Hooke's law), each of the stresses σ_{13} and σ_{23} depend only on the strain components ε_{13} and ε_{23}.
- e4) The quantities $C_{\alpha\beta\gamma3}-C_{\alpha\beta33}C_{33\gamma3}/C_{3333}$ are even in z.
- e5) The effect of the transversal stress σ_{33} is neglected in the 2D constitutive equations of the plate.

Remarks 2.1. The conditions e2) take place provided by the symmetry of the strain and stress tensors ($\varepsilon_{ij}=\varepsilon_{ji}$, $\sigma_{ij}=\sigma_{ji}$) and the hyperelasticity of the material (i.e. the strain energy does not