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An analytical approach for the mixed-mode crack in linear viscoelastic media

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ABSTRACT

In this paper, an analytical approach is developed for the fracture analysis of linear viscoelastic media. By the Laplace transform, the governing equations for the time domain (t-domain) are changed into frequency domain (s-domain). Then, a Hamiltonian system is established by introducing the dual variables of displacements and energy variational principle. In the framework of symplectic mathematics, the unknown vector consisting of displacements and stresses is expanded in terms of symplectic eigensolutions whose coefficients can be determined from the outer boundary conditions. Then t-domain solution is finally obtained by inverse Laplace transform and exact forms of fracture parameters including stress intensity factor (SIF) and J-integral are derived simultaneously. Numerical examples are provided to verify the validity of the present method. A parametric study of viscoelastic parameters and outer boundary conditions is carried out also.

1. Introduction

Viscoelastic materials such as asphalt, epoxy and solid propellants have been widely applied in many fields including civil engineering, electronic packaging, aerospace etc. (Dave et al., 2011; Duan et al., 2011; Lei et al., 2012; Lu and Wright, 1998; Wang et al., 1998). Especially, some elastic materials also appear viscoelastic property under high temperatures and pressures. Therefore, the fracture resistance associated with time of such materials is more concerned. Atkinson and Craster (1995) elaborated various topics in fracture mechanics in the inelastic materials in 20th century, the fracture parameters such as SIFs and energy release rates were all reviewed. In the past decade, a number of the theoretical works and numerical methods have been carried out by many researchers. Greenwood (2004) applied Schapery's principles (Schapery, 1975, 1984) to the particular case of a Maugis–Dugdale surface force law and a three-element viscoelastic solid. Nguyen et al. (2005) proposed a material force method to evaluate the energy release rate and work rate of dissipation for fracture in inelastic materials. Dubois and Petit (2005) developed a new path-independent integral $G\theta_v$ which allowed us to compute the energy release rate. Chen and Atkinson (2005) reduced the fracture problem of a penny-shaped crack embedded in the central layer of a composite viscoelastic material to a singular integral equation. Pitti et al. (2007, 2008) and Pitti et al. (2009) proposed a fracture algorithm uncoupling viscoelastic incremental formulation and the fracture procedure for the creep crack growth process in a viscoelastic medium. Bouchbinder and Brener (2011) calculated the scaling properties of the quasi-static energy release rate and the viscoelastic contribution to the facture energy of various biological composites, using both perturbative and non-perturbative approaches. In addition, the viscoelastic functionally graded material (FGM) recently attracted increasing attention in the field of viscoelastic fracture analysis. Jin and Paulino (2002) used the correspondence principle to study the cracks in a viscoelastic strip of a functionally graded material (FGM) under tensile loading conditions. In the same manner, Wang et al. (2014) investigated crack problem in viscoelastic FGMs with general mechanical properties. Besides, the experimental methods and numerical simulations are also effective ways to predict the fracture behaviors of viscoelastic materials. The fracture responses of viscoelastic material were examined by various techniques (Danton, 2002; Yoneyama and Takashi, 2002; Sakaue et al., 2008; Gao et al., 2011). Numerical approaches such as the finite element method (FEM) (Pan et al., 2009; Zhang et al., 2010; Dave et al., 2011; Yu and Ren, 2011; Duan et al., 2011; Lei et al., 2012) and boundary element method (BEM) (Wang and

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Nomenclature

u_i	component of displacements
σ_{ij}, ε_{ij}	component of stresses and strains
s_{ij}, e_{ij}	component of deviatoric stresses and deviatoric strains
η	coefficient of viscosity
G	shear modulus
K	bulk modulus
υ	Poisson's ratio
(x_1, x_2)	Cartesian coordinates
(r, θ)	Polar coordinates
H	Hamiltonian function
$\mathbf{q}$, $\mathbf{p}$	mutually dual vectors
$\mathbf{H}$	Hamiltonian operator matrix
μ	eigenvalue of Hamiltonian matrix
$\psi_n^{(\alpha)}$, $\psi_n^{(\beta)}$	eigensolution of α and β group
$K_{\rm I}$, $K_{\rm II}$	Mode I and II stress intensity factors
$J_{\rm int}$	J-integral

Birgisson, 2007; Syngellakis and Wu, 2008; Chen and Hwu, 2011) were introduced in this area to deal with the complicated boundary value problems. Although many achievements have been made in this area, the theoretical studies were mainly performed based on the correspondence principle between elasticity and viscoelasticity. The exact solutions of cracked viscoelastic media were limited by the complicated equations and boundary conditions. Especially, the current fracture analysis approach for viscoelastic materials belongs to the Lagrangian system with a single variable. The components of displacements and stresses cannot be obtained simultaneously. Moreover, experiments in viscoelastic materials will result in high consumption and numerical simulations cannot give the simple expression as analytical model to provide guide in this area.

So there is still a need for developing analytical method to find the exact solutions which could allow us a better understanding of the fracture behaviors. In this paper, a direct analytical symplectic approach is introduced to the fracture analysis of linear viscoelastic materials. The symplectic approach was first proposed by Zhong and his collaborators (Li et al., 2013a, 2013b; Lim and Xu, 2010; Yao et al., 2009; Zhong et al., 2009) and has been applied to the fracture mechanics (Leung et al., 2009; Zhou et al., 2013) and viscoelasticity (Xu et al., 2006). In the symplectic space, the unknown vector is composed of the displacement and stress functions and can be solved simultaneously.

The paper is organized as follows. Firstly, the basic problem is stated and the Laplace transform is taken to convert the fundamental equations from t-domain to s-domain. Secondly, the resulting equations are put into the Hamiltonian form for variable separation. The eigensolutions are found and the particular integral is obtained by the eigenfunctions expansion. Thirdly, the solution is solved and transformed back into the t-domain by using the inverse Laplace transform. Lastly, numerical examples are shown to validate the efficiency and accuracy of the symplectic approach. In addition, the discontinuous complex outer boundary conditions are included.

2. Basic problem

In the linear viscoelasticity theory, the equilibrium equations for static loading conditions, and the strain-displacement relations for the small deformations are written as

$$\sigma_{ij,j}(t) + f_i(t) = 0, \tag{1}$$

$$\varepsilon_{ij}(t) = \left[u_{i,j}(t) + u_{j,i}(t)\right]/2 \tag{2}$$

in which u_i, σ_{ij} and ε_{ij} are the displacement, stress and strain components respectively. The constitutive equations can be expressed in a differential form as

$$\begin{cases} P^K(t)\sigma_{kk}(t) = Q^K(t)\varepsilon_{kk}(t) \\ P^G(t)s_{ij}(t) = Q^G(t)e_{ij}(t) \end{cases} \tag{3}$$

where $P^K(t) = \sum_{i=0}^{m^K} p_i^K \partial^i/\partial t^i$, $Q^K(t) = \sum_{i=0}^{n^K} q_i^K \partial^i/\partial t^i$, $P^G(t) = \sum_{i=0}^{m^G} p_i^G \partial^i/\partial t^i$, $Q^G(t) = \sum_{i=0}^{n^G} q_i^G \partial^i/\partial t^i$ are the differential operator polynomials (Zhang, 2006), σ_{kk} and ε_{kk} are the volumetric stress and strain respectively, s_{ij} and e_{ij} are the deviatoric components of the stress and strain tensors. The relationships among them are

$$s_{ij} = \sigma_{ij} - \sigma_{kk}\delta_{ij}/3, \quad e_{ij} = \varepsilon_{ij} - \varepsilon_{kk}\delta_{ij}/3 \tag{4}$$

where δ_{ij} is Kronecker delta which equals to one when $i = j$ and equals to zero otherwise.

By using the Laplace transform, Eq. (3) can be transformed into the s-domain. Using the over-bar to identify the variables for the s-domain, the transformed constitutive equations are

$$\begin{cases} \overline{\sigma}_{kk}(s) = 3\overline{K}(s)\overline{\varepsilon}_{kk}(s) \\ \overline{s}_{ij}(s) = 2\overline{G}(s)\overline{e}_{ij}(s) \end{cases} \tag{5}$$

where $\overline{K}(s) = {\rm L}[K(t)] = \overline{Q}^K(s)/[3\overline{P}^K(s)]$ and $\overline{G}(s) = {\rm L}[G(t)] = \overline{Q}^G(s)/[2\overline{P}^G(s)]$ are the bulk modulus and shear modulus in the s-domain, respectively. For simplification, it can be assumed that the material behaves elastically in dilatation, i.e., $K(t) = \overline{K}(s) = K$ is constant; or the corresponding Poisson's ratio is constant and there is a relation between $\overline{G}(s)$ and $\overline{K}(s)$, i.e., $\overline{G}(s) = 3(1-2\upsilon)\overline{K}(s)/[2(1+\upsilon)]$ (Wang and Birgisson, 2007).

For the two-dimensional viscoelastic problem, the polar coordinate (r, θ) is selected such that the r-axis is along the radial direction with origin located at the central point of the circular domain in Fig. 1. The crack surfaces are along $\theta = \pm\pi$ and assumed to traction free. The equilibrium equations, strain-displacement relations and constitutive equations of the linear viscoelasticity for the s-domain can be obtained from Eqs. (1), (2) and (5), and they are

$$\begin{cases} \partial_r\overline{\sigma}_{rr} + \partial_\theta\overline{\sigma}_{r\theta}/r + (\overline{\sigma}_{rr} - \overline{\sigma}_{\theta\theta})/r + \overline{f}_r = 0 \\ \partial_\theta\overline{\sigma}_{\theta\theta}/r + \partial_r\overline{\sigma}_{r\theta} + 2\overline{\sigma}_{r\theta}/r + \overline{f}_\theta = 0 \end{cases}, \tag{6}$$

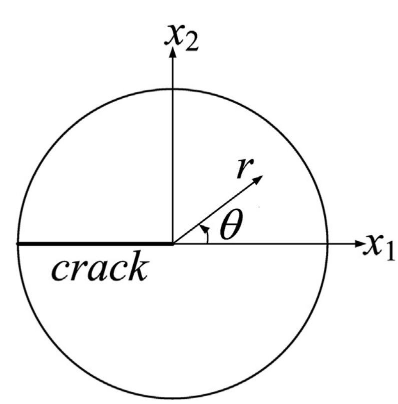

Fig. 1. The edge-cracked circular domain.