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The effects of different boundary conditions on three-dimensional cracked discs under anti-plane loading

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ABSTRACT

Accordingly to the recent multi-scale model proposed by Sih and Tang, different orders of stress singularities are related to different material dependent boundary conditions associated with the interaction between the V-notch tip and the material under the remotely applied loading conditions. This induces complex three-dimensional stress and displacement fields in the proximity of the notch tip, which are worthy of investigation.

Starting from Sih and Tang's model, in the present contribution the authors propose some analytical expressions for the calculation of the strain energy density (SED) averaged over a control volume embracing the V-notch tip. The expressions vary as a function of the different boundary conditions. Dealing with the specific crack case, the results from the analytical frame are compared with those determined numerically under linear-elastic hypotheses, by applying different constraints to the through-the-thickness crack edges in three dimensional discs subjected to Mode III loading. Free–free and free–clamped cases are considered.

Due to three-dimensional effects, the application of a nominal Mode III loading condition automatically provokes coupled Modes (I and II). Not only the intensity of the induced modes but also their degree of singularity depend on the applied conditions on the crack flanks.

The variability of local SED through the thickness of the disc is investigated by numerical analyses and compared with the theoretical trend. The capability of the SED to capture the combined three-dimensional effects is discussed in detail showing that this parameter is particularly useful when the definition of the Stress Intensity Factors (SIFs) is ambiguous or the direct comparison between SIFs with odd dimensionalities is not possible.

1. Introduction

Microscopic effects may play a significant role on the macroscopic behaviour of materials as recently highlighted by Tang and Wei (in press). This is particularly true for micro electro mechanical systems (Corigliano et al., 2011; Ardito et al., 2014) and not only under mechanical loading but also when the component is subjected to heat fluxes (Yang et al., 2014). It is therefore pertinent to have a model that can couple the microscopic effects to those at the macroscopic scale. Examined, in particular, are the inhomogeneities at the microscale arising from uneven stiffness of the material microstructures which can vary the constraints on the micro-crack. These inhomogeneities are simulated by the free–free, fixed–fixed and free–fixed boundary conditions (Tang and Sih, 2005a; 2005b; Sih and Tang, 2005). Dealing with the material microstructure an analytical multi-scale model has recently been developed by Tang and Sih (2005a). Physically, the different orders of the stress singularities are related to the different constraints associated with the defect thought as a micro V-notch at the tip of the main crack. Irregularities of the material microstructure tend to torture the crack tip being the free–clamped boundary conditions the more realistic for a general representation of what occurs on the micro V-notch (Tang and Sih, 2005a). The approach by Tang and Sih (2005a) allows to overcome under linear elastic properties the problem tied to plasticity by considering different eigenvalues at different material scales. A multiscale damage model valid for anti-plane loading has been proposed by using the singularity representation method derived for plates under in-plane extension (Tang and Sih, 2005a, 2005b). Different orders and strengths

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of singularity are uniquely associated with the boundary conditions, loadings and geometries of the defects under consideration. The stress and strain fields in the proximity of V-notch placed at the main crack tip are very complex. Some experimental and theoretical studies have been recently carried out on the fracture behaviour of V-notches making clear the interest on this topic among the scientific community (Leguillon, 2002; Murer and Leguillon, 2010; Ayatollahi et al., 2013; Torabi and Ayatollahi, 2014; Torabi and Pirhadi, 2014). The degree of complexity usually arises if the complete three-dimensional elastic problem is investigated. For this reason it is of interest to study the behavior of a V-notch under a nominal anti-plane shear loading and the induced modes automatically generated near the notch tip due to different boundary conditions in a three-dimensional component. The problem of coupled modes generated by a primary nominal applied mode has been extensively studied mainly dealing with cracks (Nakamura and Parks, 1988, 1989, Pook, 2003; Berto et al., 2011a; Kotousov et al., 2012, 2013) but also with pointed and sharply radiused V-notches (Pook, 1994, 2000, Berto et al., 2011b, 2013a; Lazzarin and Zappalorto, 2012; Zappalorto and Lazzarin, 2013). In those references it was shown that at a corner point Mode II and Mode III cannot exist in isolation. If one of these modes is applied then the other is always automatically induced. Moreover, if the free–clamped boundary conditions are applied to the edges of the notch, Mode I and II are always coupled each other also in a plane problem (Tang and Sih, 2005a,b; Berto et al., 2012). Their intensities are influenced by the shape of the notch and by the externally applied loads (Tang and Sih, 2005a). As explained above in three-dimensional pointed notches with free–clamped edges under nominal anti-plane shear loading all the Modes are locally present (Tang and Sih, 2005a; Sih and Tang, 2005). The degree of singularity of the different Modes is strongly influenced by the boundary conditions applied on the notch edges. Take for example the crack case that is generally characterized by a degree of singularity equal to 0.5 regardless of the loading mode according to Linear Elastic Fracture Mechanics. When the crack edges are characterized by free–clamped conditions different singularities can arise, depending on the considered loading Mode. This generates an odd dimensionality between the Stress Intensity Factors which are not directly comparable. Moreover there are some uncertainties in the definition of the Stress Intensity Factors on the free surfaces, due to the corner point singularity generated by the intersection of the crack with the free surfaces of the model (Benthem, 1980; Dhondt, 1998; Pook, 2013; Pook et al., 2014).

At the light of these considerations the present paper is twofold. First some analytical expressions of the strain energy density (SED) averaged over a control volume embracing the V-notch tip with a generic opening angle 2α are derived in the case of Mode III loading, as a function of different boundary conditions on the notch edges. In particular free–free and free–clamped conditions are considered in the paper combining the expressions of SED derived here for Mode III loading with those for coupled Modes obtained by extending to the specific case some previously derived expressions (Lazzarin and Zambardi, 2001).

In the second part of the paper the specific case of three-dimensional cracked discs characterized by different thicknesses is investigated numerically. Free–free and free–clamped boundary conditions are applied to the FE models. The complex stress field is accurately investigated and the SED through the thickness of the disc obtained by numerical analyses is compared with the theoretical trend derived by using the developed analytical frame. Due to the uncertainties in the definition of the Stress Intensity Factors on the free surfaces, mentioned above, and the odd dimensionality of the SIFs due to the applied boundary conditions the strain energy averaged in a control volume (SED) is employed in the present contribution to quantify the stress intensity through the thickness of the disc. For a review of the SED the reader can refer to a recent paper (Berto and Lazzarin, 2014). This parameter has been successfully used by Lazzarin and co-authors to assess the fracture strength of a large bulk of materials, characterized by different control volumes, subjected to wide combinations of static loading (Berto et al., 2013b; Lazzarin et al., 2014) and the fatigue strength of welded joints (Radaj et al., 2009a; 2009b) and notched components (Berto and Lazzarin, 2011; Berto et al., 2011c). As shown by Lazzarin et al. (2008, 2010) an intrinsic advantage of the SED approach is that it permits automatically to take into account higher order terms and it is easy to calculate by using coarse meshes.

In the present paper a careful comparison is carried out between free–free and free–clamped constraint conditions along the edges of the notch. Although the only aim of this paper is to investigate the crack case, as visible from the developed analytical frame, the SED can be easily applied also to point V-notches and a direct comparison can be directly drawn between the crack and the V-notch cases. This will be the subject of future works.

2. Analytical framework: stress and displacement fields under Mode III

This section summarises the analytical frame giving the expressions of the singular stress and displacement fields in the proximity of the V-notch tip. A Mode III loading condition is treated, by varying the boundary conditions on the notch edges. An isotropic and homogeneous material under linear elastic conditions is taken into account.

2.1. Free–free

In the presence of a sharp V-notch with stress free surfaces, named the free–free condition hereafter, the stress distributions due to anti-plane loading (Mode III) are (Qian and Hasebe, 1997; Dunn et al., 1997; Tang and Sih, 2005b):

$$\sigma_{\mathrm{rz}} = \frac{K_{3,\mathrm{F-F}}}{\sqrt{2\pi}} 2\lambda_3 \mathrm{r}^{\lambda_3-1}[\sin(\lambda_3\theta)] \tag{1a}$$

$$\sigma_{\theta\mathrm{z}} = \frac{K_{3,\mathrm{F-F}}}{\sqrt{2\pi}} 2\lambda_3 \mathrm{r}^{\lambda_3-1}[\cos(\lambda_3\theta)] \tag{1b}$$

The only displacement component different from zero is:

$$\mathrm{u_z} = \frac{K_{3,\mathrm{F-F}}}{G}\sqrt{\frac{2}{\pi}}\mathrm{r}^{\lambda_3}[\sin(\lambda_3\theta)] = 2(1+\nu)\frac{K_{3,\mathrm{F-F}}}{E}\sqrt{\frac{2}{\pi}}\mathrm{r}^{\lambda_3}[\sin(\lambda_3\theta)] \tag{2}$$

where G is the shear modulus, E the Young's modulus and ν the Poisson's ratio.

The stress and displacement fields are given in a cylindrical coordinate system centered at the notch tip (Fig. 1); r is the radial coordinate, θ is the angle measured from the notch bisector line and z is the out-of-plane coordinate.

The parameter $K_{3,\mathrm{F-F}}$ represents the Mode III Notch Stress Intensity Factor (NSIF), thought of as the natural extension of the Mode I and Mode II NSIFs first defined by Gross and Mendelson (1972). The degree of stress singularity depends on the Mode III eigenvalue λ_3, which varies as a function of the notch opening angle 2α (Seweryn and Molski, 1996; Tang and Sih, 2005b). Under the