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The non-singular Green tensor of gradient anisotropic elasticity of Helmholtz type

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ABSTRACT

In this paper, we derive the three-dimensional Green tensor of the theory of gradient anisotropic elasticity of Helmholtz type, a particular version of Mindlin's theory of gradient elasticity with only one characteristic parameter. In contrast with classical anisotropic elasticity, it is found that in gradient anisotropic elasticity of Helmholtz type both the Green tensor and its gradient are non-singular at the origin. On the other hand, the Green tensor rapidly converges to its classical counterpart a few characteristic lengths away from the origin. Therefore, the Green tensor of gradient anisotropic elasticity of Helmholtz type can be used as a physically-based regularization of the classical anisotropic Green tensor. Using the non-singular Green tensor, the Kelvin problem is studied in the framework of gradient anisotropic elasticity.

1. Introduction

The concept of Green function was first introduced in the context of potential theory by Green (1828), as an auxiliary tool to solve linear inhomogeneous partial differential equations (PDEs). Loosely speaking, the Green function of a given linear PDE represents its particular solution corresponding to a concentrated source acting at the origin. The Green function is often referred to as the "fundamental solution" of the PDE, because solutions corresponding to other source fields can be obtained from the Green function by convolution. For this reason, Green functions are mathematical objects of essential importance for the solution of many problems in physics and engineering. Moreover, the concept of Green function constitutes the basis of important numerical methods for boundary value problems, such as the boundary element method (e.g. Becker, 1992).

In classical linear elasticity, the governing PDE is defined by the Navier differential operator, and its Green function is a tensor-valued function of rank two, also known as the Green tensor. When contracted with a concentrated force acting at the origin, the Green tensor yields the displacement field in the infinite elastic medium. The Green tensor of the isotropic Navier operator was first derived in closed form by Lord Kelvin (1882). For anisotropic materials, Lifshitz and Rosenzweig (1947) and Synge (1957) were able to derive the Green tensor in terms of a one-dimensional integral on the unit circle. Barnett (1972) extended the result to the derivatives of the Green tensor, and showed that the line-integral representation is well suited for numerical integration (see also Bacon et al., 1979; Teodosiu, 1982; Balluffi, 2012).

The Green tensor of the Navier operator plays an important role in the nano- and micro-mechanics of materials, and in particular in the elastic theory of defects such as cracks, dislocations and inclusions (Mura, 1987). In this context, the Green tensor and its first derivatives are essential for the evaluation of the elastic fields and interaction energies of defects. However, practical applications in nano-mechanics encounter difficulties due to the presence of singularities at the origin in both the Green tensor and in its derivatives. The emergence of these unphysical singularities can ultimately be ascribed to the lack of characteristic length parameters in the classical theory of elasticity, which prevents the possibility of capturing size effects inevitably associated with the discrete nature of matter.

In the framework of Mindlin's gradient theory of elasticity (Mindlin, 1964), characteristic length scale parameters are inherently incorporated into the formulation due to the dependence of the strain energy density on strain gradients. To fix an order of

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magnitude, it is worth mentioning that characteristic length scale parameters of first strain gradient elasticity are in the order of ~10^{-10} m for several fcc and bcc metals (Shodja et al., 2013). Therefore, gradient elasticity becomes relevant to nano-mechanical phenomena at such length scales. From a mathematical viewpoint, the resulting higher-order PDEs provide a physical-based regularization of the classical singularities (see, e.g. Lazar, 2014). Polyzos et al. (2003); Gao and Ma (2009) and Lazar (2012, 2013) found the non-singular isotropic Green tensor in a simplified version of Mindlin's gradient elasticity containing only one characteristic length. We shall refer to this simplified framework as gradient elasticity of Helmholtz type (Lazar and Maugin, 2005; Lazar et al., 2005; Lazar, 2014), because the regularization function is given by the Green function of the Helmholtz operator.

The aim of this paper is to derive the non-singular Green tensor of gradient elasticity of Helmholtz type in the anisotropic case. This tensor can be regarded as a physical-based regularization of the classical (singular) Green tensor of anisotropic elasticity, to be used in problems of nano-mechanics of materials whenever the effects of anisotropy are of interest. In this regard, we emphasize that, because only one gradient parameter is used, the regularization function is isotropic. Thus, the theory of gradient anisotropic elasticity of Helmholtz type is anisotropic in the elastic material moduli (bulk anisotropy), but the weak non-locality possesses an isotropic character.

This paper is organized as follows. In Section 2, we summarize the framework of gradient anisotropic elasticity of Helmholtz type. In Section 3, we construct the Green tensor of the anisotropic Helmholtz–Navier operator governing the theory. Using the Fourier transform method, the non-singular anisotropic Green tensor is obtained as a surface integral. In addition, we derive expressions for the gradient of the Green tensor. Moreover, we show that the singular Green tensor of classical anisotropic elasticity can be obtained in the limit of vanishing characteristic length. As a special case, we also derive the closed-form expression of the non-singular Green tensor of gradient isotropic elasticity. In Section 4, we implement the main equations of the paper and apply them to the Kelvin problem in gradient anisotropic elasticity to illustrate the effects of non-singular anisotropy. Finally, discussion and conclusions are presented in Section 5. The derivation of the theory of gradient anisotropic elasticity from Mindlin's theory of anisotropic gradient elasticity is given in Appendix A. In Appendix B, some technical details are given.

2. Gradient anisotropic elasticity of Helmholtz type

In order to introduce the framework of gradient anisotropic elasticity of Helmholtz type, we consider an infinite elastic body in three-dimensional space and assume that the gradient of the displacement field $\boldsymbol{u}$ is additively decomposed into an elastic distortion tensor $\boldsymbol{\beta}$ and an inelastic[1] distortion tensor $\boldsymbol{\beta}^*$:

$$\partial_j u_i = \beta_{ij} + \beta^*_{ij}. \tag{1}$$

In terms of the elastic strain tensor $\boldsymbol{e}$

$$e_{ij} = \frac{1}{2}\left(\beta_{ij} + \beta_{ji}\right), \tag{2}$$

the strain energy density of the body is assumed to be (see Appendix A)

$$\mathscr{W} = \frac{1}{2}C_{ijkl}e_{ij}e_{kl} + \frac{1}{2}\ell^2 C_{ijkl}\partial_m e_{ij}\partial_m e_{kl}, \tag{3}$$

where ℓ is a characteristic length scale parameter, and C_{ijkl} is the anisotropic tensor of elastic moduli possessing the following symmetry properties:

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}. \tag{4}$$

The quantities conjugate to the elastic strain tensor and its gradient are the Cauchy stress tensor $\boldsymbol{\sigma}$ and the double stress tensor $\boldsymbol{\tau}$, respectively. These are defined as:

$$\sigma_{ij} = \frac{\partial \mathscr{W}}{\partial e_{ij}} = C_{ijkl}e_{kl}, \tag{5}$$

$$\tau_{ijk} = \frac{\partial \mathscr{W}}{\partial(\partial_k e_{ij})} = \ell^2 C_{ijmn}\partial_k e_{mn} = \ell^2 \partial_k \sigma_{ij}. \tag{6}$$

In the presence of a body forces density $\boldsymbol{b}$, the static Lagrangian density of the system becomes:

$$\mathscr{L} = -\mathscr{W} - \mathscr{V} = -\left(\frac{1}{2}C_{ijkl}\beta_{ij}\beta_{kl} + \frac{1}{2}\ell^2 C_{ijkl}\partial_m\beta_{ij}\partial_m\beta_{kl}\right) + u_i b_i, \tag{7}$$

where

$$\mathscr{V} = -u_i b_i \tag{8}$$

is the potential of the body force.

The condition of static equilibrium is expressed by the Euler–Lagrange equations obtained vanishing the directional derivative of the Lagrangian with respect to a variation of the displacement field. In gradient elasticity this reads (e.g. Maugin, 1993; Agiasofitou and Lazar, 2009):

$$\frac{\delta\mathscr{L}}{\delta u_i} = \frac{\partial\mathscr{L}}{\partial u_i} - \partial_j\frac{\partial\mathscr{L}}{\partial(\partial_j u_i)} + \partial_k\partial_j\frac{\partial\mathscr{L}}{\partial(\partial_k\partial_j u_i)} = 0. \tag{9}$$

In terms of the Cauchy and double stress tensors, Eq. (9) takes the following form (Mindlin, 1964):

$$\partial_j\left(\sigma_{ij} - \partial_k\tau_{ijk}\right) + b_i = 0. \tag{10}$$

Using the last equality in Eq. (6), one can see that in gradient elasticity of Helmholtz type (Lazar, 2013, 2014), Eq. (10) simplifies to.

$$L\partial_j\sigma_{ij} + b_i = 0, \tag{11}$$

where

$$L = 1 - \ell^2\Delta \tag{12}$$

is the Helmholtz operator. Using Eqs. (1) and (5), Eq. (11) can be cast in the following Helmholtz–Navier equation for displacements:

$$LL_{ik}u_k + f_i = 0, \tag{13}$$

where

$$L_{ik} = C_{ijkl}\partial_j\partial_l \tag{14}$$

is the anisotropic Navier operator, and

$$f_i = b_i - C_{ijkl}\partial_j L\beta^*_{kl} \tag{15}$$

[1] The inelastic distortion comprises plastic and thermal effects, and is typically an incompatible field. When the inelastic distortion is absent the elastic distortion is compatible.