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A nth-order shear deformation theory for the bending analysis on the functionally graded plates

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A B S T R A C T

This paper focus on the bending analysis of functionally graded plates by a nth-order shear deformation theory and meshless global collocation method based on the thin plate spline radial basis function. Reddy's third-order theory can be considered as a special case of present nth-order theory ($n = 3$). The governing equations are derived by the principle of virtual work. The displacement and stress of a simply supported functionally graded plate under sinusoidal load are calculated to verify the accuracy and efficiency of the present theory.

1. Introduction

In recent years, functionally graded materials had been utilized in the aerospace and other industries because of their superior heat-shielding properties. The functionally graded material for high-temperature applications may be composed of ceramic and metal. The material properties of functionally graded material vary continuously along certain dimension of the structure, but that of the fiber-reinforced laminated composite materials are discontinuous across adjoining layers which result in the delaminating mode of failure.

Many researchers have studied the behaviors of functionally graded plates. Vel and Batra (2004) presented the three-dimensional exact solution for the vibration of functionally graded rectangular plates. Ferreira et al. (2005) studied the static characteristics of functionally graded plates using third-order shear deformation theory and a meshless method based on the multiquadrics radial basis function. Ferreira et al. (2006) calculated the natural frequencies of functionally graded plates by the multiquadrics radial basis function. Zenkour (2006) proposed a generalized shear deformation theory for bending analysis of functionally graded plates. Ferreira et al. (2007) studied the static deformations of functionally graded plates using the radial basis function collocation method and a higher-order shear deformation theory. They select the shape parameter in the radial basis functions by an optimization procedure based on the cross-validation technique. Carrera et al. (2008) presented the static analysis of functionally graded material plates subjected to transverse mechanical loadings. The unified formulation and principle of virtual displacements were employed to obtain both closed-form and finite element solutions. Matsunaga (2008) calculated the natural frequencies and buckling stresses of plates made of functionally graded materials (FGMs) using a 2-D higher-order deformation theory. Carrera et al. (2011) evaluated the effect of thickness stretching in plate/shell structures made by materials which are functionally graded (FGM) in the thickness directions. Xiang et al. (2011) proposed an n-order shear deformation theory for free vibration of functionally graded and composite sandwich plates.

In recent years, the various higher-order shear deformation theories were proposed to analyze the plates. Touratier (1991) presented a standard plate theory which accounts for cosine shear stress distribution and free boundary conditions for shear stress upon the top and bottom surfaces of the plate. Soldatos (1992) presented a general two-dimensional theory suitable for the static and/or dynamic analysis of a transverse shear deformable plate, constructed of a homogeneous, monoclinic, linearly elastic material and subjected to any type of shear tractions at its lateral plane. Karama et al. (2003) presented

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a new multi-layer laminated composite structure model to predict the mechanical behaviour of multi-layered laminated composite structures. They introduced an exponential function as the shear stress function. Reddy (1984) developed a higher-order shear deformation theory which accounts for parabolic distribution of the transverse shear strains through the thickness of the laminated plate. Aydogdu (2009) proposed a new higher-order laminated composite plate theory in which a new shear stress function was used.

In this paper, an n-order shear deformation theory is used to analyze the static characteristics of functionally graded plates. The present n-order shear deformation theory satisfies the zero transverse shear stress boundary conditions on the top and bottom surface of the plate. The third-order theory of Reddy can be considered as a special case of present n-order theory ($n = 3$). Displacement and stress of the simply supported functionally graded plate under sinusoidal load are computed by present n-order theory and a meshless global collocation method based on the thin plate spline radial basis function. The results are compared with the available published results.

2. The governing equations based on the *n*th-order shear deformation theory

The displacement field of the n-order shear deformation theory is

$$\begin{aligned} U &= u(x,y) + z\phi_x(x,y) - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\phi_x(x,y) + \frac{\partial w(x,y)}{\partial x}\right) \\ V &= v(x,y) + z\phi_y(x,y) - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\phi_y(x,y) + \frac{\partial w(x,y)}{\partial y}\right), \\ &\quad n = 3,5,7,9\ldots \\ W &= w(x,y) \end{aligned} \tag{1}$$

where u, v, w, ϕ_x and ϕ_y are the unknown displacement functions. h is the thickness of the plate.

The strain can be expressed in the form of

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + z\frac{\partial \phi_x}{\partial x} - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ \varepsilon_y &= \frac{\partial v}{\partial y} + z\frac{\partial \phi_y}{\partial y} - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} \\ &\quad z^n \left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) \\ \gamma_{yz} &= \left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\phi_y + \frac{\partial w}{\partial y}\right) \\ \gamma_{xz} &= \left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\phi_x + \frac{\partial w}{\partial x}\right) \end{aligned} \tag{2}$$

We obtain the following Euler–Lagrange equations using the dynamic version of the principle of virtual displacements

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ &\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \\ &\frac{\partial Q_x}{\partial x} - C_2\frac{\partial R_x}{\partial x} + \frac{\partial Q_y}{\partial y} - C_2\frac{\partial R_y}{\partial y} + C_1\left(\frac{\partial^2 P_x}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_y}{\partial y^2}\right) = q \\ &\frac{\partial M_x}{\partial x} - C_1\frac{\partial P_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - C_1\frac{\partial P_{xy}}{\partial y} - Q_x + C_2 R_x = 0 \\ &\frac{\partial M_{xy}}{\partial x} - C_1\frac{\partial P_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - C_1\frac{\partial P_y}{\partial y} - Q_y + C_2 R_y = 0 \end{aligned} \tag{3}$$

where $C_1 = \frac{1}{n}\left(\frac{2}{h}\right)^{n-1}$, $C_2 = (2/h)^{n-1}$.

$$\begin{aligned} N_{\alpha\beta} &= \int_{-h/2}^{h/2} \sigma_{\alpha\beta}\,dz,\ M_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta} z\,dz,\ P_{\alpha\beta} \\ &= \int_{-h/2}^{h/2} \sigma_{\alpha\beta} z^n\,dz,\ Q_\alpha = \int_{-h/2}^{h/2} \sigma_{\alpha z}\,dz,\ R_\alpha \\ &= \int_{-h/2}^{h/2} \sigma_{\alpha z} z^{n-1}\,dz, \quad (\alpha, \beta = x, y) \end{aligned} \tag{4}$$

The stress–strain relationships of the functionally graded plate in the global x–y–z coordinate system can be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{bmatrix} Q_{11}(z) & Q_{12}(z) & 0 & 0 & 0 \\ Q_{12}(z) & Q_{11}(z) & 0 & 0 & 0 \\ 0 & 0 & Q_{66}(z) & 0 & 0 \\ 0 & 0 & 0 & Q_{66}(z) & 0 \\ 0 & 0 & 0 & 0 & Q_{66}(z) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{5}$$

where

$$Q_{11}(z) = \frac{E(z)}{1-\mu^2},\ Q_{12}(z) = \frac{\mu E(z)}{1-\mu^2},\ Q_{66}(z) = \frac{E(z)}{2(1+\mu)} \tag{6}$$

In the Eq. (6), μ is the Poisson's ratio, the variation of Young's modulus E is given as:

$$E(z) = (E_c - E_m)\left(\frac{1}{2} + \frac{z}{h}\right)^p + E_m \tag{7}$$

where E_c and E_m denote the elasticity modulus of the ceramic and metal, respectively. p is power law index. z is the distance from mid-plane. h is the thickness of the plate. As can be seen Eq. (7), $E(z) = E_c$ at the top surface $z/h = 0.5$, and $E(z) = E_m$ at the bottom surface $z/h = -0.5$. Top surface of functionally graded plate is pure ceramic, and bottom surface is pure metal.

Substituting Eq. (2) and Eq. (5) into Eq. (4), the resultants of functionally graded plate can be expressed in terms of displacement as follows