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Hamiltonian structure for the Charney–Hasegawa–Mima equation in the asymptotic model regime

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Abstract

We show that the Charney–Hasegawa–Mima (CHM) equation in the asymptotic model (CHM-AM) regime possesses the non-canonical Hamiltonian structure. The CHM-AM corresponds to the CHM equation in the asymptotic limit of length scales large compared to the Rossby deformation radius. It is shown that the Hamiltonian structure of the CHM-AM cannot be derived directly from that of the CHM equation by taking a simple limit of a length scale. Both the two-dimensional (2-D) Euler equation and the CHM-AM are regarded as special cases of a generalized 2-D fluid system, the so-called α-turbulence system. The existence of the Hamiltonian structure of the CHM-AM obtained in this study and that of the 2-D Euler equation implies the existence of the Hamiltonian structure of the α-turbulence system.

Keywords: Charney–Hasegawa–Mima equation; Non-canonical Hamiltonian fluid dynamics; Casimir invariants; Noether's theorem; Geophysical fluid dynamics; α-turbulence

1. Introduction

Large-scale extratropical atmospheric and oceanic fluid motions are approximately in the state of geostrophic balance (the Colioris force is balanced by the pressure gradient). The Charney–Hasegawa–Mima (CHM) equation, which governs the nearly geostrophic motion of a shallow homogeneous fluid

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layer with a free surface, is given by

$$\frac{\partial}{\partial t}(\nabla^2\psi - \lambda^2\psi) + \partial(\psi, \nabla^2\psi) = 0, \tag{1}$$

where $\psi(x, y, t)$ is a stream function, $\partial(A, B) \equiv \partial_x A \partial_y B - \partial_y A \partial_x B$ is the Jacobian operator, $\nabla^2 \equiv \partial_{xx} + \partial_{yy}$ is the horizontal Laplacian, $\lambda = L/R_{\mathrm{d}}$, L is a typical horizontal scale of the flow, R_{d} is the Rossby deformation radius.

In the limit of $\lambda \to 0$, (1) becomes the two-dimensional (2-D) Euler equation. When $\lambda \to \infty$, replacing the time t by $\lambda^2 t$, (1) reduces to

$$\frac{\partial}{\partial t}\psi + \partial(\nabla^2\psi, \psi) = 0. \tag{2}$$

Since (2) has been called the asymptotic model (Larichev and McWilliams, 1991), we will call (2) the CHM equation in the asymptotic model regime (CHM-AM). Eq. (2) governs the nearly geostrophic flow whose horizontal scale is larger than the deformation radius and is a simple model for the large-scale dynamics of the extratropical ocean and Jovian atmosphere, where there exists motions that involve scales exceeding the internal deformation radius.

In this paper, we present the non-canonical Hamiltonian structure of the CHM-AM. Although the Hamiltonian representation of the CHM equation (1) was already presented (Weinstein, 1983), that of the CHM-AM has not been yet. The result of the present work shows that one cannot obtain the Hamiltonian structure of the CHM-AM directly from that of the CHM equation by taking a simple limit $\lambda \to \infty$ (see, Section 3).

2. Brief review of the non-canonical Hamiltonian formalism

In general, the Eulerian variables that describe the fluid are non-canonical. The non-canonical Hamiltonian formalism (see, for example, Scinocca and Shepherd, 1992) requires that the governing equations can be cast in the symplectic form

$$\frac{\partial}{\partial t}\boldsymbol{q} = \boldsymbol{J}\frac{\delta\mathscr{H}}{\delta\boldsymbol{q}}, \tag{3}$$

where $\boldsymbol{q}$ is a column vector of the dependent field variables which are functions of space and time defined over some spatial domain D, $\mathscr{H}$ is a conserved functional of $\boldsymbol{q}$ and called Hamiltonian, $\boldsymbol{J}$ is a matrix operator, and $\delta\mathscr{H}/\delta\boldsymbol{q}$ is a functional derivatives of $\mathscr{H}$ with respect to $\boldsymbol{q}$. $\delta\mathscr{H}/\delta\boldsymbol{q}$ is defined by

$$\delta\mathscr{H}[\boldsymbol{q}; \delta\boldsymbol{q}] \equiv \lim_{\varepsilon\to 0}\frac{\mathscr{H}[\boldsymbol{q}+\varepsilon\delta\boldsymbol{q}] - \mathscr{H}[\boldsymbol{q}]}{\varepsilon} \equiv \int_D \left(\frac{\delta\mathscr{H}}{\delta\boldsymbol{q}}\right)^{\top}\delta\boldsymbol{q}\,\mathrm{d}\boldsymbol{x}, \tag{4}$$

where $(*)^{\top}$ is the transpose of $(*)$ and $\delta\mathscr{H}$ is called the first variation of $\mathscr{H}$. In addition, the Poisson bracket, which is defined by

$$[\mathscr{F}, \mathscr{G}] \equiv \int_D \left(\frac{\delta\mathscr{F}}{\delta\boldsymbol{q}}\right)^{\top}\boldsymbol{J}\frac{\delta\mathscr{G}}{\delta\boldsymbol{q}}\,\mathrm{d}\boldsymbol{x}, \tag{5}$$