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Free vibration analysis of elastic elliptic cylinders with an eccentric elliptic cavity

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ABSTRACT

An exact elastodynamic model based on Navier equations of linear elasticity is formulated to describe the three-dimensional natural oscillations of an elliptic cylinder of finite length with shear diaphragm end conditions, and containing an inner (coaxial) elliptical cavity of arbitrary size, location, and orientation. The formulation is based on Helmholtz decomposition theorem, the method of separation of variables in elliptical coordinates, and the translational addition theorems for Mathieu functions. The first three natural frequencies are calculated for selected cylinder lengths, cross-sectional aspect ratios, and cavity location/orientation parameters. Also, some representative 3D deformation mode shapes are depicted in vivid graphical form. The precision of solutions is checked through proper convergence studies, and the validity of results is verified with the aid of a commercial finite element package as well as by comparison with the available literature data. The presented exact Mathieu series solution is believed to be the first attempt on the vibrational characteristics of finite-length eccentric elliptical cylinders.

1. Introduction

Dealing with three dimensional dynamic problems of elasticity is an intricate matter with many complications linked to the associated governing differential equations and satisfaction of specified boundary conditions. In particular, the exact elastodynamic analyses of solid and thick hollow cylinders of finite lengths are difficult structural dynamic problems that have recently been thoroughly investigated [1–4]. Analytical solutions can readily be obtained only when the outer and inner cylinder boundaries are concentric circles or confocal ellipses [2,5,6]. If the boundaries are eccentric or of some other shape, semi-analytical approaches as well as a wide range of approximate numerical methods have generally been utilized to treat the problem [7–9]. In contrast, there are very few analytical research works that address the dynamic behavior of eccentric cylinders [3,10–13] or cylindrical shells with circumferential wall thickness variations [14,15]. For example, Suzuki and Leissa [14,15] employed thin and thick shell theories along with power series expansion method to study free vibrational characteristics of circular and noncircular (elliptical) cylindrical shells of circumferentially varying thickness with shear diaphragm end conditions. More recently, Hasheminejad and Mirzaei [3] employed the translational addition theorem for cylindrical wave functions to develop an exact 3D elasticity series solution for free vibration analysis of a simply-supported circular hollow cylinder of finite length with an eccentric inner circular cavity. Just recently, Hasheminejad and Mousavi-Akbarzadeh [12,13] extended the latter work [3] to address steady-state and transient acoustic radiation from simply-supported eccentric hollow circular cylinders.

The main purpose of current paper is to employ the method of separation of variables in elliptical coordinates and the translational addition theorems for Mathieu functions to generalize the work done in Ref. [3] and obtain an exact series solution for free vibrations of an elastic elliptic cylinder with shear diaphragm end conditions, and containing an arbitrarily-located coaxial elliptical cavity (see Fig. 1). The eccentric cylindrical components are extensively used as the basic structural elements in a wide variety of industrial and physical applications [3,16,17]. Thus, a comprehensive dynamic characterization of such structures will provide a real basis for the design engineer in assessing the suitability of introducing cavities in these elements at each situation. The proposed model is of both academic and technical interest due to its inherent value as a canonical problem in structural dynamics. It can particularly complement experimental procedures in structural parameter identification, and characterization/control of structural non-uniformities [18]. Lastly, the set of highly accurate converged solutions can not only reveal the physical characteristics of the problem but also serve as the benchmark in assessment of highly restrictive numerical or asymptotic approaches [19–21].

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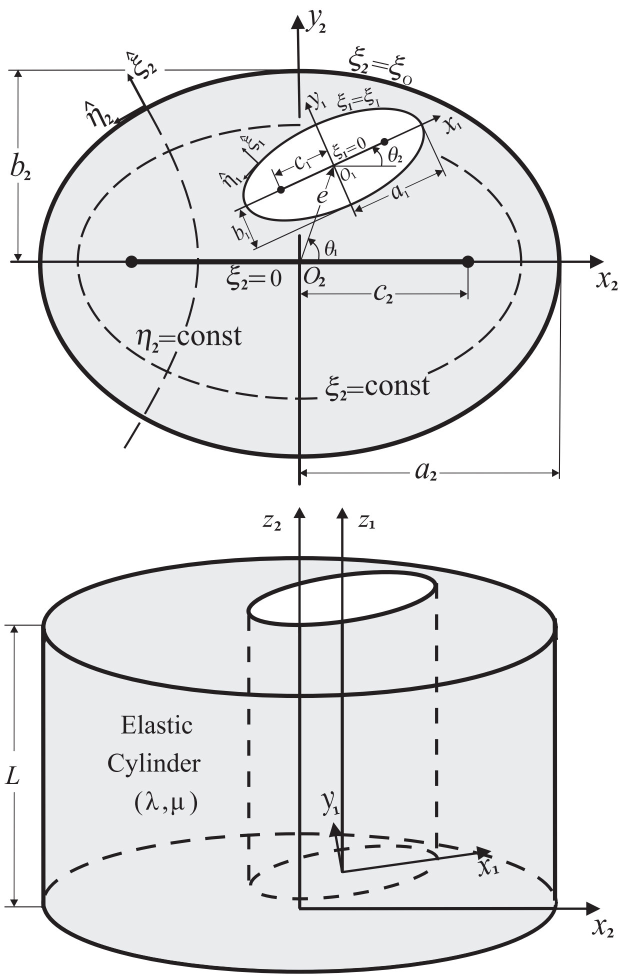

Fig. 1. Problem geometry.

2. Formulation

2.1. Basic governing equations and field expansions

The cylinder is supposed to be fabricated of linearly elastic homogeneous, and isotropic material with the classic stress–strain relation given as [22]

$$\sigma_{ij}=\lambda\delta_{ij}\varepsilon_{kk}+2\mu\varepsilon_{ij}, \tag{1}$$

where (λ,μ) are Lame constants, and δ_{ij} is Kronecker delta function. Also, in the absence of body forces, the displacement field is governed by the Navier's equation [23]:

$$(\lambda+2\mu)\nabla(\nabla\bullet\mathbf{u})-\mu\nabla\times(\nabla\times\mathbf{u})=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{2}$$

where ρ is the material density, and the vector displacement $\mathbf{u}$ can advantageously be decomposed into the gradient of a scalar potential along with the curl of a vector potential as

$$\mathbf{u}=\nabla\varphi+\nabla\times\boldsymbol{\psi}, \tag{3}$$

where the vector potential $\boldsymbol{\psi}$ is defined in terms of the scalar shear wave potential pairs (ψ,χ) through the classical relation [22–24]:

$$\boldsymbol{\psi}=\nabla\times\psi\mathbf{e}_z+\nabla\times(\nabla\times\chi\mathbf{e}_z),$$

which automatically satisfies the zero divergence condition $\nabla\bullet\boldsymbol{\psi}=0$ (where, $\mathbf{e}_z$, is the unit vector in the z-direction, pointing along the cylinder axis). The above (Helmholtz) decomposition (3) allows one to manipulate the dynamic equation of motion (2) into the following fully-uncoupled scalar wave equations

$$c_p^2\nabla^2\varphi=\ddot{\varphi},\qquad c_s^2\nabla^2\psi=\ddot{\psi},\qquad c_s^2\nabla^2\chi=\ddot{\chi}, \tag{4}$$

where $c_p^2=(\lambda+2\mu)/\rho$ and $c_s^2=\mu/\rho$ are the dilatational and distortional wave speeds in the elastic medium, respectively.

Taking advantage of the general elliptic cylindrical coordinate system (ξ,η,z), the displacement components, $\mathbf{u}=(U_\xi,U_\eta,U_z)$, may be written in terms of the compressional and shear wave potentials as [24]:

$$U_\xi=\frac{1}{h}\left(\frac{\partial\varphi}{\partial\xi}+\frac{\partial^2\psi}{\partial\xi\partial z}-\frac{1}{c_s^2}\frac{\partial^3\chi}{\partial t^2\partial\eta}\right),$$
$$U_\eta=\frac{1}{h}\left(\frac{\partial\varphi}{\partial\eta}+\frac{\partial^2\psi}{\partial\eta\partial z}+\frac{1}{c_s^2}\frac{\partial^3\chi}{\partial t^2\partial\xi}\right),$$
$$U_z=\frac{\partial\varphi}{\partial z}-\frac{1}{c_s^2}\frac{\partial^2\psi}{\partial t^2}+\frac{\partial^2\psi}{\partial z^2}, \tag{5}$$

and with the relevant stress–displacement relations given as [22]:

$$\sigma_{\xi\xi}=\left(\frac{\lambda+2\mu}{h}\right)\frac{\partial U_\xi}{\partial\xi}+\frac{\lambda}{h}\frac{\partial U_\eta}{\partial\eta}+\lambda\frac{\partial U_z}{\partial z}+\frac{\lambda c^2\sinh(2\xi)}{2h^3}U_\xi+\left(\frac{\lambda+2\mu}{2h^3}\right)c^2\sin(2\eta)U_\eta,$$
$$\sigma_{\xi\eta}=\frac{\mu}{h}\left[\frac{\partial U_\xi}{\partial\eta}+\frac{\partial U_\eta}{\partial\xi}-\frac{c^2\sinh(2\xi)}{2h^2}U_\eta-\frac{c^2\sin(2\eta)}{2h^2}U_\xi\right],$$
$$\sigma_{\xi z}=\mu\left(\frac{\partial U_\xi}{\partial z}+\frac{1}{h}\frac{\partial U_z}{\partial\xi}\right),$$
$$\sigma_{zz}=\frac{\lambda}{h}\frac{\partial U_\xi}{\partial\xi}+\frac{\lambda}{h}\frac{\partial U_\eta}{\partial\eta}+(\lambda+2\mu)\frac{\partial U_z}{\partial z}+\frac{\lambda c^2\sinh(2\xi)}{2h^3}U_\xi+\left(\frac{\lambda+2\mu}{2h^3}\right)c^2\sin(2\eta)U_\eta, \tag{6}$$

where $h^2=c^2\left[\sinh^2(\xi)+\sin^2(\eta)\right]$ is the scale factor.

Now, consider a finite-length elliptical cylinder of the aspect ratio a_2/b_2 with the associated elliptic coordinate system (ξ_2,η_2,z_2), and containing an arbitrarily located elliptical cavity of aspect ratio a_1/b_1 with the associated elliptic coordinate system (ξ_1,η_1,z_1), where the triad (e,θ_1,θ_2) denotes the cavity location/orientation, as depicted in Fig. 1. The foci of the inner/outer elliptical boundaries $\xi_{1,2}=\xi_{1,O}$ are located at the coordinates $x_j=\pm c_j=\pm\left(a_j^2-b_j^2\right)^{1/2}, y_j=0\ (j=1,2)$ in the reference frame $O_jx_jy_jz_j$, with the associated semi-major and semi-minor axes $a_{1,2}=c_{1,2}\cosh(\xi_{1,O})$, and $b_{1,2}=c_{1,2}\sinh(\xi_{1,O})$. Assuming harmonic time variations, one can recast the uncoupled wave Eq. (4), in the (ξ_1,η_1,z) coordinate system as [4]:

$$\left(\frac{\partial^2\varphi}{\partial\xi_1^2}+\frac{\partial^2\varphi}{\partial\eta_1^2}\right)+2q_{p1}\left[\cosh(2\xi_1)-\cos(2\eta_1)\right]\varphi=0,$$
$$\left(\frac{\partial^2\psi}{\partial\xi_1^2}+\frac{\partial^2\psi}{\partial\eta_1^2}\right)+2q_{s1}\left[\cosh(2\xi_1)-\cos(2\eta_1)\right]\psi=0,$$
$$\left(\frac{\partial^2\chi}{\partial\xi_1^2}+\frac{\partial^2\chi}{\partial\eta_1^2}\right)+2q_{s1}\left[\cosh(2\xi_1)-\cos(2\eta_1)\right]\chi=0, \tag{7}$$

where $q_{p1}=k_p^2c_1^2/4$, $q_{s1}=k_s^2c_1^2/4$ and $k_p^2=\omega^2/c_p^2-\gamma^2$, $k_s^2=\omega^2/c_s^2-\gamma^2$, are the longitudinal and transverse wave numbers, ω is the frequency, and γ is the separation constant. Consequently, the general displacement potential solutions in the finite cylindrical domain may be expanded as proper combinations of even and odd ordinary Mathieu functions of first kind (i.e., ce_n, se_n) alongside even and odd modified Mathieu functions of first and second kinds (i.e., Je_n, Jo_n,