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Determination of a time-dependent coefficient in the bioheat equation

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ABSTRACT

In this paper, the identification of the time-dependent blood perfusion coefficient in the bioheat equation is considered as an inverse heat source problem with nonlocal boundary and integral energy over-determination conditions. The boundary element method (BEM) based on using the fundamental solution for the heat equation is employed, together with either the second-order Tikhonov regularization combined with finite differences, or with a smoothing spline regularization technique for computing the first-order derivative of a noisy function. A couple of benchmark numerical examples are presented to verify the accuracy and stability of the solution.

1. Introduction

The bioheat equation establishes a mathematical connection between the tissue temperature and the arterial blood perfusion which are the dominant components in human physiology, see Trucu et al. [17]. It involves a blood perfusion coefficient whose determination is of much interest [15].

In this paper, we consider the determination of the unknown time-dependent blood perfusion coefficient for the bioheat equation under nonlocal boundary and integral conditions. We mention that time-dependent coefficient identification problems with nonlocal boundary and/or integral overdetermination conditions have recently attracted revitalizing interest, e.g. the reconstruction of a time-dependent diffusivity [10], a blood perfusion coefficient [8], or a heat source [9,4]. A simple transformation is used to reduce the bioheat equation to the classical heat equation. This inverse problem has already been proved to be uniquely solvable in Kerimov and Ismailov [11], but no numerical reconstruction has been attempted. Therefore, the purpose of this study is to devise a numerical stable method for obtaining the solution of the inverse problem.

2. Mathematical formulation

Let us consider the inverse problem consisting of finding the time-dependent blood perfusion coefficient function $P(t)$ and the temperature of the tissue $u(x,t)$, i.e. the pair $(P(t), u(x,t))$, from the class $C[0,T] \times (C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T))$, where $D_T = \{(x,t) | 0 < x < 1, 0 < t \le T\} = (0,1) \times (0,T]$, $T > 0$ is given, satisfying the one-dimensional time-dependent bioheat equation [16]

$$u_t(x,t) = u_{xx}(x,t) - P(t)u(x,t) + f(x,t), \quad (x,t) \in D_T, \tag{1}$$

where f is a known heat source term, subject to the following initial and boundary conditions:

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{2}$$

$$-u_x(0,t) = \alpha u(0,t), \quad t \in [0,T], \tag{3}$$

$$u(0,t) = u(1,t), \quad t \in [0,T], \tag{4}$$

$$\int_0^1 u(x,t)\, dx = E(t), \quad t \in [0,T], \tag{5}$$

where the function φ is given and it denotes the initial temperature, α is a given constant heat transfer coefficient, and E represents the mass or energy of the system. Note that the nonlocal periodic boundary condition (4) is encountered in biological applications [14], whilst the mass/energy specification (5) models processes related to particle diffusion in turbulent plasma [7], or heat conduction [1]. The physical constraint that the blood perfusion $P(t)$ is positive can also be imposed [12].

Note that the case $\alpha = 0$ has been dealt with in [9]. Herein, we consider the case $\alpha \neq 0$ whose unique solvability and local continuous dependence of the solution upon the data of the inverse problem (1)–(5) have been established in [11]. Moreover, the global continuous dependence of the solution upon the data can also be established based on a Gronwall's-type inequality [6].

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Consider now the following transformation [2]:

$$v(x,t) = r(t)u(x,t). \tag{6}$$

Then the inverse problem (1)–(5) becomes

$$v_t = v_{xx} + r(t)f(x,t), \quad (x,t) \in D_T, \tag{7}$$

$$v(x,0) = \varphi(x), \quad x \in [0,t], \tag{8}$$

$$v(0,t) = v(1,t), \quad v_x(0,t) + \alpha v(0,t) = 0, \quad t \in [0,T], \tag{9}$$

with the transformed integral condition

$$\int_0^1 v(x,t)\,dx = E(t)r(t), \quad t \in [0,T]. \tag{10}$$

We also have that $r \in C^1[0,T]$, $r(0)=1$, $r(t)>0$, for $t \in [0,T]$. Solving the inverse problem (7)–(10) for the solution pair $(r(t), v(x,t))$ yields afterwards the solution pair $(P(t), u(x,t))$ for the inverse problem (1)–(5) as given by

$$P(t) = \frac{r'(t)}{r(t)} \quad \text{and} \quad u(x,t) = \frac{v(x,t)}{r(t)}, \quad (x,t) \in D_T. \tag{11}$$

From Eq. (11) one can observe that the ill-posedness of the inverse problem consists of the numerical differentiation of the noisy function $r(t)$ which would need regularization.

3. The boundary element method (BEM)

In this section, we apply the BEM to the one-dimensional inverse problem (7)–(10), in order to approximate the solution $(r(t), v(x,t))$ which in turn, via (11), leads to the original solution $(P(t), u(x,t))$ of the inverse problem (1)–(5). Utilizing the BEM is classical with the use of the fundamental solution for the heat equation and Green's identities. The fundamental solution for the heat equation (7) is given by

$$G(x,t,y,\tau) = \frac{H(t-\tau)}{\sqrt{4\pi(t-\tau)}} \exp\left(-\frac{(x-y)^2}{4(t-\tau)}\right),$$

where H is the Heaviside step function. By applying this fundamental solution and Green's formula to the heat equation (7) recast this as the boundary integral equation:

$$\begin{aligned}\eta(x)v(x,t) = &\int_0^t \left[G(x,t,\xi,\tau)\frac{\partial v}{\partial n(\xi)}(\xi,\tau) - v(\xi,\tau)\frac{\partial G}{\partial n(\xi)}(x,t,\xi,\tau)\right]_{\xi\in\{0,1\}} d\tau \\ &+ \int_0^1 G(x,t,y,0)v(y,0)\,dy \\ &+ \int_0^1\int_0^T G(x,t,y,\tau)r(\tau)f(y,\tau)\,d\tau\,dy, \quad (x,t) \in [0,1]\times(0,T],\end{aligned} \tag{12}$$

where $\eta(0)=\eta(1)=\frac{1}{2}$, $\eta(x)=1$ for $x \in (0,1)$, and $\underline{n}$ is the outward unit normal to the space boundary $\{0,1\}$. The boundaries $\{0\}\times[0,T]$ and $\{1\}\times[0,T]$ are divided into N small time-intervals $[t_{j-1}, t_j]$, $j=\overline{1,N}$, with $t_j = jT/N$, $j=\overline{0,N}$, whilst the initial domain $[0,1]\times\{0\}$ is divided into N_0 small cells $[x_{k-1}, x_k]$, $k=\overline{1,N_0}$ with $x_k = k/N_0$, $k=\overline{0,N_0}$. Using a piecewise constant BEM we assume that

$$v(0,t) = v(0,\tilde{t}_j) =: h_{0j}, \quad v(1,t) = v(1,\tilde{t}_j) =: h_{1j},$$
$$\frac{\partial v}{\partial n}(0,t) = \frac{\partial v}{\partial n}(0,\tilde{t}_j) =: q_{0j}, \quad \frac{\partial v}{\partial n}(1,t) = \frac{\partial v}{\partial n}(1,\tilde{t}_j) =: q_{1j},$$
$$v(x,0) = v(\tilde{x}_k,0) = \varphi(\tilde{x}_k) =: \varphi_k, \quad \text{for} \quad t \in (t_{j-1}, t_j], x \in [x_{k-1}, x_k],$$

where $\tilde{t}_j = (t_{j-1}+t_j)/2$ and $\tilde{x}_k = (x_{k-1}+x_k)/2$, $j=\overline{1,N}$, $k=\overline{1,N_0}$. For the source term, the functions $f(x,t)$ and $r(t)$ are approximated to be the piecewise constant functions:

$$f(x,t) = f(x,\tilde{t}_j), \quad r(t) = r(\tilde{t}_j) =: r_j \quad \text{for } t \in (t_{j-1}, t_j].$$

Then the integral equation (12) can be approximated as

$$\begin{aligned}\eta(x)v(x,t) = &\sum_{j=1}^{N}\left[A_{0j}(x,t)q_{0j} + A_{1j}(x,t)q_{1j} - B_{0j}(x,t)h_{0j} - B_{1j}(x,t)h_{1j}\right] \\ &+ \sum_{k=1}^{N_0} C_k(x,t)\varphi_k + \sum_{j=1}^{N} D_j(x,t)r_j, \quad (x,t) \in [0,1]\times(0,T],\end{aligned} \tag{13}$$

where the coefficients and the double integral source term are given by

$$A_{\xi j}(x,t) = \int_{t_{j-1}}^{t_j} G(x,t,\xi,\tau)\,d\tau,$$
$$B_{\xi j}(x,t) = \int_{t_{j-1}}^{t_j} \frac{\partial G}{\partial n(\xi)}(x,t,\xi,\tau)\,d\tau, \quad \xi = \{0,1\}, \tag{14}$$

$$C_k(x,t) = \int_{x_{k-1}}^{x_k} G(x,t,y,0)\,dy, \quad D_j(x,t) = \int_0^1 f(y,\tilde{t}_j)A_{yj}(x,t)\,dy. \tag{15}$$

Note that the first three integrals in (14) and (15) can be evaluated analytically [3], whereas the integral source term $D_j(x,t)$ is approximated by Simpson's rule of integration. Applying Eq. (13) at the boundary nodes $(0,\tilde{t}_i)$ and $(1,\tilde{t}_i)$ for $i=\overline{1,N}$ yields the system of $2N$ linear equations

$$A\underline{q} - B\underline{h} + C\underline{\varphi} + D\underline{r} = \underline{0}, \tag{16}$$

where

$$A = \begin{bmatrix} A_{0j}(0,\tilde{t}_i) & A_{1j}(0,\tilde{t}_i) \\ A_{0j}(1,\tilde{t}_i) & A_{1j}(1,\tilde{t}_i)\end{bmatrix}_{2N\times 2N},$$
$$B = \begin{bmatrix} B_{0j}(0,\tilde{t}_i)+\frac{1}{2}\delta_{ij} & B_{1j}(0,\tilde{t}_i) \\ B_{0j}(1,\tilde{t}_i) & B_{1j}(1,\tilde{t}_i)+\frac{1}{2}\delta_{ij}\end{bmatrix}_{2N\times 2N},$$

$$C = \begin{bmatrix} C_k(0,\tilde{t}_i) \\ C_k(1,\tilde{t}_i)\end{bmatrix}_{2N\times N_0}, \quad D = \begin{bmatrix} D_j(0,\tilde{t}_i) \\ D_j(1,\tilde{t}_i)\end{bmatrix}_{2N\times N},$$
$$\underline{q} = \begin{bmatrix} q_{0j} \\ q_{1j}\end{bmatrix}_{2N}, \quad \underline{h} = \begin{bmatrix} h_{0j} \\ h_{1j}\end{bmatrix}_{2N},$$

$\underline{\varphi} = [\varphi_k]_{N_0}$, $\underline{r} = [r_j]_N$, and δ_{ij} is the Kronecker delta symbol. We can also collocate (9) and (10) as

$$h_{0j} = h_{1j}, \quad q_{0j} = \alpha h_{0j}, \quad j=\overline{1,N}, \tag{17}$$

$$e_i r_i = \frac{1}{N_0}\sum_{k=1}^{N_0} v(\tilde{x}_k, \tilde{t}_i), \quad i=\overline{1,N}, \tag{18}$$

where $e_i := E(\tilde{t}_i)$. Using (13) and (18) yields

$$\frac{1}{N_0}\sum_{k=1}^{N_0}[A_k^I\underline{q} - B_k^I\underline{h} + C_k^I\underline{\varphi} + D_k^I\underline{r}] = W\underline{r}, \tag{19}$$

where

$$A_k^I = [A_{0j}(\tilde{x}_k,\tilde{t}_i)\ A_{1j}(\tilde{x}_k,\tilde{t}_i)]_{N\times 2N}, \quad B_k^I = [B_{0j}(\tilde{x}_k,\tilde{t}_i)\ B_{1j}(\tilde{x}_k,\tilde{t}_i)]_{N\times 2N},$$

$$C_k^I = [C_k(\tilde{x}_k,\tilde{t}_i)]_{N\times N_0}, \quad D_k^I = [D_j(\tilde{x}_k,\tilde{t}_i)]_{N\times N}, \quad W = \mathrm{diag}(e_1,\ldots,e_N).$$

Eliminating $\underline{q}$ and $\underline{h}$ from (16), (17) and (19) yields a linear system of N equations

$$X\underline{r} = \underline{y}, \tag{20}$$

with N unknowns, where

$$X = \frac{1}{N_0}\sum_{k=1}^{N_0}\left[-\left(A_k^I - \frac{1}{\alpha}(B_k^I + B_k^I*)\right)\left(A - \frac{1}{\alpha}(B+B^*)\right)^{-1} D + D_k^I\right] - W,$$

$$\underline{y} = \frac{1}{N_0}\sum_{k=1}^{N_0}\left[\left(A_k^I - \frac{1}{\alpha}(B_k^I + B_k^I*)\right)\left(A - \frac{1}{\alpha}(B+B^*)\right)^{-1} C - C_k^I\right]\underline{\varphi},$$