International Journal of Mechanical Sciences 78 (2014) 91–96

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

Limit analysis based on GM criterion for defect-free pipe elbow under internal pressure

Shunhu Zhang [a,*], Xiaonan Wang [a], Binna Song [a], Dewen Zhao [b]

[a] *Shagang School of Iron and Steel, Soochow University, Suzhou 215021, China*
[b] *State Key Lab of Rolling and Automation, Northeastern University, Shenyang 110819, China*

ARTICLE INFO

Article history:
Received 5 March 2012
Received in revised form
22 August 2013
Accepted 21 October 2013
Available online 8 November 2013

Keywords:
Limit analysis
Pipe elbow
Internal pressure
Strain hardening exponent
GM criterion

ABSTRACT

Analytical solution of limit load for a defect-free pipe elbow is obtained under internal pressure using GM (geometric midline), in which the strain hardening effect has been taken into account. The limit load is a function of ratio of thickness to radius t_0/r_0, strain hardening exponent n, curvature influence factor m and ultimate tensile strength. Comparison with FE and analytical results of other investigators was performed. Although the limit loads calculated by GM criterion are little higher than the traditional analytical results, the GM results are in good agreement with FE results. Besides, the effect of different criteria, strain hardening exponent, ratio of thickness to radius, as well as curvature influence factor on the limit loads are also discussed systematically.

1. Introduction

Pressure pipeline mainly includes straight pipe, pipe elbow and tee joint. The pipe elbow is an important part of pipeline and has been widely used in industrial pipe, utility pipe and transmission pipeline. Since the pipe elbow must be designed to avoid a collapse, its limit load must be accurately estimated to ensure the reliability of the piping systems during service. Pipe elbow is more complicated than straight pipe, due to the existence of its bend radius and angle.

Determining the limit load of elastic–plastic structure, there are two main methods. One is based on elastic–plastic deformation theory with the assumption that when a structure undergoes from elastic state to plastic state its load-carrying ability will decrease and ultimately be lost at plastic limit state. Applying this method to solve engineering problem, people often be confronted with mathematical difficulties, and few research results in this aspect have been demonstrated. The other method is based on plastic deformation law, in which the material is assumed to be rigid-plasticity material and characteristics can only be studied in plastic limit state.

The analysis of the latter method is simple because the elastic deformation doesn't need to be taken into consideration. However, other authentic characteristics of the material, such as the distribution of stress and strain prior to entering plasticity state, cannot be reflected [1–5].

Miller [6] summarized existing limit load solutions for pipe elbow, but also noted that these solutions are lower bounds and should be used with caution. Experimental works on pipe elbow are limited. Griffiths [7] performed experimental study on both cracked and uncracked elbows mainly to the see the effect of cracks on limit loads. For uncracked elbow, he suggested to multiply the Calladine formula by a factor of 1.33 to account for the stiffening effect of tangent pipes attached to the elbow. Hilsenkopf et al. [8] carried out 25 tests on various elbows under in-plain (opening and closing) and out-of-plane bending moment. Influence of pressure, temperature and cyclic loading on the deformation behavior was studied. It was concluded that in-plane closing bending reduced the stiffness of the elbow and was the most penalizing loading mode. It was also concluded that internal pressure stiffened the elbow and resisted the ovalization. Based on detailed three-dimensional (3-D) FE limit analysis, Kim and Oh [9] provided plastic limit, collapse and instability load solutions for pipe elbow under combined pressure and bending, in which the effects of pipe elbow geometries were accurately reflected. Kim et al. [10] also performed three-dimensional elastic–plastic finite analyses on locally wall-thinned elbows subjected to in-plane bending with a constant internal pressure, and the effects of wall-thinning parameters, such as the thinning depth, length, circumferential angle, and location, and the bend on the collapse behavior of wall-thinned elbows were also investigated. Shalaby and Younan [11] presented ductile failure surfaces for a

---

[*] Corresponding author. Tel./fax: +86 512 671 60032.
E-mail address: shzhang@suda.com (S. Zhang).

0020-7403/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijmecsci.2013.10.022

Nomenclature

t_0	original wall thickness
r_0	original average radius
n	strain hardening exponent
m	curvature influence factor
σ'_{uts}	ultimate tensile stress
σ_s	yield stress
σ_f	limit stress, $==(\sigma_s+\sigma'_{uts})/2$
p	internal pressure
r	mean radius
t	mean wall thickness
R_0	neutral line curvature radius
θ	radial bend angle
φ	circumferential intersection angle
A_φ	infinitesimal section area
F_φ	hoop internal force
e	2.7183
w	yield criterion dependent constant
p_{limit}	limit load of a pipe elbow
p_0	limit load of a straight pipe
λ	bend characteristic, $=R_0t/r^2$

range of isolated 90° pipe elbow subject to combined internal pressure and closing in-plane moment. In the analysis, an ideal elastic-perfectly plastic material and large deformation theory were assumed, and consequently different behavior was found for closing moment. Chattopadhyay et al. [12] calculated collapse loads for a range 90° bends with attached straight pipes under combined pressure and in-plane bending loads using the NISA 3D finite element program [13]. The piping system was modeled using 3D finite solid finite elements and included strain hardening and large deformation effects. Without question, finite element analysis is the effective method to investigate limit loads of pipe elbows, especially to the complicated situation of combined loads. However, few analytical solutions for pipe elbows under an acceptability criterion and considering the effect of strain hardening on the limit load are available [14–17].

In the present paper, the assumption that the material analyzed is elastic–plastic material is made, and the material obeys the pure power–law curve. With a linear GM (geometric midline) criterion, the analytical solution of the limit load of a defect-free pipe elbow under internal pressure is obtained, in which the strain hardening effect has been taken into account. The results of the analytical solution are compared with analytical and FEM results of other researchers. What's more, the effects of different yield criteria, initial ratio of thickness to radius t_0/r_0, strain hardening exponent n and curvature influence factor m on limit load are also discussed.

2. GM criterion

The GM (geometric midline) of error triangle $B'FB$ between Tresca and TSS (Twin Shear Stress) yield loci [18] in the π-plane were linked together and used as the yield locus of new yield criterion, which was called geometrical midline yield (GM) criterion [19]. The GM criterion locus on the π-plane is equilateral and non-equiangular dodecagon which is much closer to Mises locus, as shown in Fig. 1.

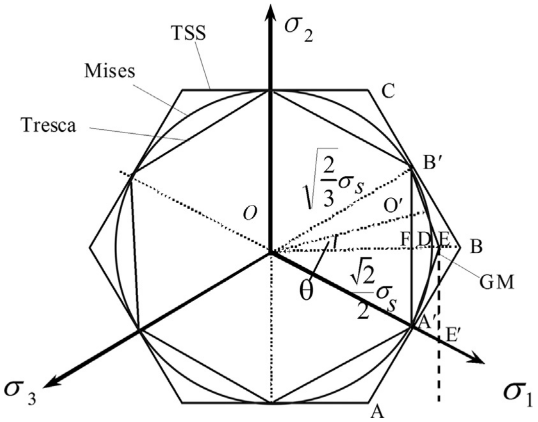

Fig. 1. GM locus in the π-plane.

The GM criterion has been used in metal forming, calculation of crack tip spreading area and so on [20,21]. With the convention $\sigma_1 \ge \sigma_2 \ge \sigma_3$, then the equations of GM criterion in the Haigh Westergaard stress space are as follows:

$$\sigma_1-\frac{2}{7}\sigma_2-\frac{5}{7}\sigma_3=\sigma_s \quad \text{if } \sigma_2 \le \frac{1}{2}(\sigma_1+\sigma_3) \tag{1a}$$

$$\frac{5}{7}\sigma_1+\frac{2}{7}\sigma_2-\sigma_3=\sigma_s \quad \text{if } \sigma_2 \ge \frac{1}{2}(\sigma_1+\sigma_3) \tag{1b}$$

where σ_s is material yield stress.

3. Stress field for pipe elbow

The stress state of pipe elbow under internal pressure p is different from that of straight pipe. A thin-wall pipe elbow with mean radius r, wall thickness t, neutral-line curvature radius R_0, radial bend angle θ and internal pressure p is shown in Fig. 2.

The hoop stress of pipe elbow section is shown in Fig. 3, in which the angle φ is circumferential intersection angle formed by radius and neutral axis. Relative to the point C, the angel φ is positive if rotation is anticlockwise. At point Q, a differential element of pipe wall with angle of $d\varphi$ on the radical section is intercepted and its circumferential arc length can be expressed by $r\ d\varphi$. In Fig. 2, the radical arc length of infinitesimal section at φ angle can be expressed by $(R_0+r\ \sin\ \varphi)\ d\theta$. Therefore, the infinitesimal section area A_φ formed by wall thickness t and arc length $(R_0+r\ \sin\ \varphi)\ d\theta$ is as follows:

$$A_\varphi=t(R_0+r\ \sin\ \varphi)\ d\theta \tag{2}$$

The stress σ_φ normal to the section A_φ is defined as the hoop stress, and then the corresponding hoop internal force F_φ is as

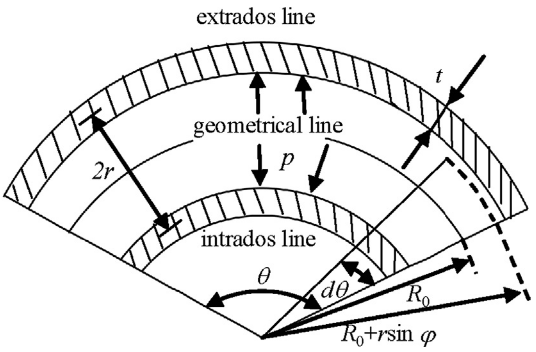

Fig. 2. Pipe elbow under inner pressure.