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Acoustic eigenproblems of elliptical cylindrical cavities with multiple elliptical cylinders by using the collocation multipole method

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ABSTRACT

Acoustic eigenproblems of 3-D elliptical cylindrical cavities having multiple elliptical cylinders are solved by means of a 3-D semi-analytical formulation based on a multipole expansion, directional derivative, collocation technique and singular value decomposition (SVD). The multipole expansion for the acoustic pressure is formulated in terms of angular and radial Mathieu functions due to elliptical boundaries considered here. The boundary conditions are satisfied by uniformly collocating points along the boundaries. When considering sound-hard or Neumann boundary conditions, the normal derivative of acoustic pressure with respect to non-local elliptic coordinates is derived using the directional derivative for multiply-connected domain problems. By truncating the multipole expansion, a finite linear algebraic system is acquired. The direct searching approach is applied to identify the natural frequencies by using the SVD. Several numerical examples are examined, including those of an elliptical cylindrical cavity, a confocal elliptical annulus cylindrical cavity and an elliptical cylindrical cavity with two elliptical cylinders. Convergence and comparison studies are done to demonstrate the validation and accuracy of the present method. No spurious eigensolutions are found in the proposed formulation. Excellent accuracy and fast rate of convergence are the main features of the present method thanks to its semi-analytical character.

1. Introduction

Eigensolutions of three-dimensional (henceforth, denoted as 3-D) acoustic cavities, *i.e.* natural frequency and natural mode, can give valuable information when designing cavities related to mechanical systems such as automobile mufflers and hermetic compressors [1], since the occurrence of resonance peaks in the transmission loss characteristics of a muffler can be explained by means of these data. Although numerical methods, such as the finite element method (FEM) and boundary element method (BEM) [2], can solve these problems, analytical solutions, if available, usually result in accurate and fast rate convergence methodologies and give a physical insight into the considered problem. Thus a semi-analytical approach to eigenproblems for 3-D acoustic cavities is presented in this work.

Over the last few decades, there is a great difference in the number of works to analytically solve eigenproblems for simple geometries such as circular or annular cavities [3–6] and a similar analysis for elliptical cylindrical cavities. However, except Ref. [1], little attention has been focused on acoustic problems involving elliptical boundaries, probably because it involves the ill-familiar and complicated Mathieu functions [7–10] in comparison with the relatively simpler Bessel functions for the case of a circular cylindrical geometry. Based on a 3-D analytical formulation, Hong and Kim [1] derived the characteristic equations of both hollow and annular elliptical cylindrical cavities in terms of Mathieu functions and presented the circular cylinders as a special case of their results. Chen *et al.* [5,6] applied the null-field boundary integral equation method (BIEM) to solve 2-D eigenproblems with circular boundaries. Recently, Chen *et al.* extend the BIEM to deal with eigenproblems of a confocal elliptical membrane [11], performing an analytical investigation of spurious eigenvalues and providing several remedies to suppress them. It is well known that the BIEM (or BEM) belongs to the boundary-type method which can reduce the dimension of the original problem by one order. Consequently the number of the unknowns is much less than that of the domain type methods such as the FEM. In addition, the domain mesh generation is not required, which is generally a difficult and time consuming task. However, the singular or hypersingular integration in the formulation usually makes the BEM difficult to implement. Furthermore, spurious eigenvalues always occur for multiply-connected problems even though they can be suppressed by using complex kernel functions for simply-connected domain problems. Although many methods have been proposed to solve these problems, including the degenerate kernel functions [6], regularization technique [12] and various

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algorithms [5] that suppress spurious or fictitious eigenvalues, additional techniques inevitably lead to tedious formulations and complex calculations, limiting their practical application. Therefore, a regular (*i.e.* non-singular) method that is free of spurious or fictitious eigenvalues is needed.

The multipole method for solving multiply-connected domain problems was firstly devised by Zàviška [13], and applied to the interaction of waves with arrays of circular cylinders by Linton and Evans [14]. The addition theorem is frequently applied to transform the multipole expansion into one of coordinate systems for satisfying the specified boundary conditions. For a circular boundary, some applications, including the water wave scattering problem [14], free vibration problem of circular membranes [15], free vibration problem of circular plates [16] and the flexural wave scattering problem [17], have been available in the literature. Furthermore, Chatjigeorgiou and Mavrakos [18,19] proposed an analytical approach to the hydrodynamic diffraction by multiple elliptical cylinders using the addition theorem of the Mathieu functions. This procedure is mathematically elegant. However, one must deal with the complex formulation and associated numerical calculations due to the infinite series form of the addition theorem so that its applications are limited.

Instead of using the addition theorem, this work presents a collocation multipole method to solve eigenproblems of 3-D acoustic cavities with a multiply connected domain. The acoustic field within the cavity is expanded in series of Mathieu functions to satisfy the Helmholtz equation in the elliptic cylindrical coordinates. The normal derivatives of acoustic pressure with respect to non-local elliptical coordinates are exactly calculated (*i.e.* without truncated error) by using the directional derivative to satisfy the Neumann boundary conditions (or sound-hard conditions). By uniformly collocating points on boundaries and truncating the multipole expansion, a finite system of simultaneous linear algebraic equations is derived. Based on the direct searching approach [20], nontrivial eigenvalues can be identified by detecting the zero determinant of a linear system using the SVD technique. Based on the corresponding eigenvectors (*i.e.* coefficients of the multipole expansion), the corresponding natural modes can be acquired. Several numerical examples are examined, including those of an elliptical cylindrical cavity, a confocal elliptical annulus cylindrical cavity and an elliptical cylindrical cavity with two elliptical cylinders. The proposed results are compared with those obtained by available analytical solutions and the FEM software program ABAQUS [21]. Finally the existence of spurious eigenvalues via the proposed method is also examined.

2. The 3-D Helmholtz equation and its general solution in the elliptic cylindrical coordinate system

A 3-D elliptical cylindrical cavity of length L containing H nonoverlapping elliptical cylinders has a domain Ω whose boundaries are $z=0$, $z=L$ and

$$B = \cup_{j=1}^{Q} B_j. \tag{1}$$

Fig. 1 shows its cross-section, where O_j and B_j denote the center of the jth ellipse and its boundary, the subscript $j=1, \ldots, Q$ ($Q=1+H$ and O_1 is the center of the outer ellipse). We will use $Q+1$ observer coordinate systems: (x, y, z) is a global Cartesian coordinate system centered at O; (ξ_j, η_j, z_j), $j=1,\ldots, Q$ is the jth local elliptic cylindrical coordinate system centered at O_j with global Cartesian coordinates (x^j, y^j, z^j) and the local coordinate system makes an angle θ_j with respect to the global coordinate system. The major and minor axes of jth ellipse are $2a_j$ and $2b_j$, respectively. The wave equation for the acoustic pressure $P(\boldsymbol{x}, t)$ is

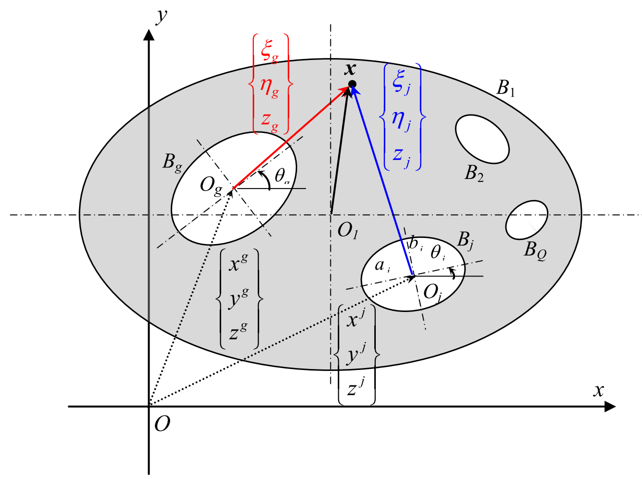

Fig. 1. The cross-section of a three-dimensional elliptical cylindrical cavity with H elliptical cylinders.

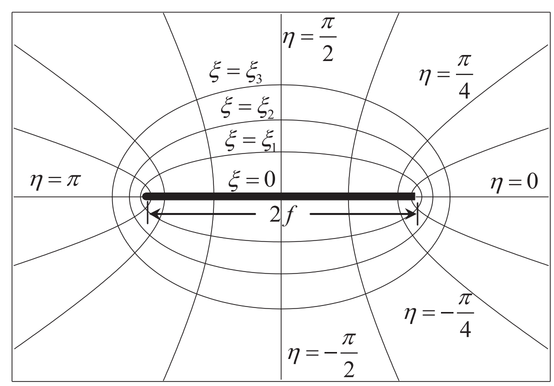

Fig. 2. Elliptical coordinate system

$$\nabla^2 P(\boldsymbol{x}, t) - \frac{\partial^2 P(\boldsymbol{x}, t)}{c^2 \partial t^2} = 0, \tag{2}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator, c is the speed of sound in the acoustic medium and $\boldsymbol{x}$ is a typical field point in the domain Ω. For the natural mode analysis, the solution of Eq. (2) is given by

$$P(\boldsymbol{x}, t) = p(\boldsymbol{x}) e^{-i\omega t}, \tag{3}$$

where ω is the radian frequency. Hence the complex-valued function $p(\boldsymbol{x})$ satisfies the following 3-D Helmholtz equation,

$$(\nabla^2 + k^2) p(\boldsymbol{x}) = 0, \tag{4}$$

where $k = \omega/c$ is the wave number.

To properly deal with the geometry considered here, the elliptic cylindrical coordinate system [7–10] as shown in Fig. 2 should be used. The elliptic coordinates (ξ, η, z) are related to the rectangular coordinates (x, y, z) by the following relations

$$\begin{aligned} x + iy &= f \cosh(\xi + i\eta), \\ z &= z, \end{aligned} \tag{5}$$

where $i = \sqrt{-1}$, ξ is a radial coordinate $(\xi \ge 0)$, η is an angular coordinate $(0 \le \eta < 2\pi)$ and $2f$ is the interfocal distance. Separating and equating real and imaginary parts of Eq. (5) gives

$$\begin{aligned} x &= f \cosh \xi \cos \eta, \\ y &= f \sinh \xi \sin \eta, \\ z &= z. \end{aligned} \tag{6}$$