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Upper bound limit analysis using radial point interpolation meshless method and nonlinear programming

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ABSTRACT

This paper presents a numerical upper bound limit analysis using radial point interpolation method (RPIM) and a direct iterative method with nonlinear programming. By expressing the internal plastic dissipation power with a kinematically admissible velocity field obtained through RPIM interpolation, the upper bound problem is formulated mathematically as a nonlinear programming subjected to single equality constraint which is solved by a direct iterative method. To evaluate the integration of internal power dissipation rate without any background integral cell, a new meshless integration technique based on Cartesian Transformation Method (CTM) is employed to transform the domain integration first as boundary integration and then one-dimensional integration. The effectiveness and accuracy of the proposed approach are demonstrated by two classical limit analysis problems. Further discussion is devoted to optimal selection of relevant parameters for the computation.

1. Introduction

As a proved direct and efficient approach to estimate the ultimate bearing capacity for structures, limit analysis has long been used in the design of a wide range of applications in civil and geotechnical engineering, such as shells, plates, foundations, retaining walls and slopes. Relying frequently on hand calculation in early days, limit analysis has now been dominated by numerical solutions with the aid of modern computers. Almost all engineering structures, no matter how complex their shapes and/or loading conditions might be, can now be conveniently discretized by numerical methods, such as Finite Element Method (FEM). Both the lower bound and the upper bound theorem can be reformulated as numerical optimization problems, and be applied to the discretized physical domain in sought for limit loads.

Finite elements and linear programming have commonly been used for numerical limit analysis for long (see, e.g., [51,1,20,6,17,63,64,71,62]). With the recent progress in the theory of nonlinear programming (hereafter shortened as NLP), a wide variety of advanced numerical techniques have been developed in limit analysis and more rigorous solutions can be sought. Typical examples include the constrained nonlinear optimization formulation based on mixed finite elements developed by Zouain et al. [75] for cohesive materials, and its recent generalization to frictional-cohesive materials by [49,50]. Recently, more advanced nonlinear programming techniques such as those based on the primal-dual interior point method [32,58–61] and those based on the second-order cone programming (SOCP) [52–55,56,34,19,37,38] have also been successfully applied to the limit analysis involving different materials.

Upper bound limit analysis has traditionally been based on finite element method for both purely cohesive materials and cohesive-frictional materials. The plastic incompressible condition in the analysis can be typically treated by such techniques as discontinuous velocity field [62,49,50], penalty function method [47,43], mixed formulations [18,2,9] and variational principle [68]. High-order elements [71,53] and cell-based smoothing finite element method [45,37] have also been developed to overcome the issue of volumetric locking when the penalty function method is used. In treating non-differential plastic dissipation function in numerical upper bound limit analysis, a wide range approaches including viscous plastic regularization [30,10], smoothed method ([2,3,23,69,29], etc.) and direct iterative method based on distinguished rigid and plastic regions [72,73,47,8,40–43] have been employed.

More recently, meshless methods have received much attention in applications relevant to numerical limit analysis. For example, Chen et al. [13] and Le et al. [36] have developed a lower bound approach using element-free Galerkin (EFG) approach with moving least squares (MLS) method to construct the self-equilibrium stress basis vectors and the static admissible stress field. Le et al. [35] have also developed an upper bound limit analysis approach based on EFG method, in which the MLS

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approximation is employed to construct the kinematically admissible velocity field. The upper bound limit analysis is then formulated as a SOCP problem and solved by a primal-dual interior point method originally proposed by Andersen et al. [3]. Moreover, as an extension of limit analysis, shakedown analysis of structures and solids with repeated loads can also be performed based on EFG [14] and meshless local Petrov-Galerkin (MLPG) method [15]. Without requiring the discretization of physical domain into meshes, meshless methods have been proved to offer improved computational efficiency over FEM with reasonable accuracy.

There are two key issues deciding whether or not an element-free Galerkin method can be successfully applied to this upper bound limit analysis. The first is pertinent to how the essential boundary conditions can be effectively reinforced, and the second is related to accurate numerical integration of the plastic dissipation power. First, it is well-known that in EFG method the shape function $\Phi_i(\mathbf{x}_j)$ lacks the property of Kronecker delta function, i.e. $\Phi_i(\mathbf{x}_j) \neq \delta_{ij}$, where δ_{ij} is the Kronecker delta function. It is hence difficult to ensure that the approximation of nodal displacement $u^h(\mathbf{x}_i)$ is exactly equal to the fictious nodal values $\hat{u}_i$ at node $\mathbf{x}_i$, i.e., $u^h(\mathbf{x}_i) = \sum \Phi_i(\mathbf{x}_j)\hat{u}_i \neq \hat{u}_i$. Consequently, the displacement boundary conditions cannot be directly enforced, i.e., $\hat{u}_b \neq \bar{u}$, where $\hat{u}_b$ is the fictious nodal value at boundary node $\mathbf{x}_b$ and $\bar{u}$ is the prescribed displacement. We notice that Le et al. [35] have adopted a collocation method proposed by Zhu and Atluri [74] to treat the boundary conditions. This method, however, may lead to increasing constraints for the NLP problem. Second, numerical integration of dissipation function has traditionally been performed by using either nodal integration method (see, e.g. [4]), or the Gauss quadrature based on an integral background cell (see, [16]). Chen et al. [12,11] have also developed a stabilized conforming nodal integration (SCNI) which proves to be robust but needs a voronoi cell. Various issues regarding accuracy and efficiency still need to be tackled with the various methods.

This paper presents a study using EFG method for limit analysis, in an attempt to improve its performance in the above two aspects. A novel numerical procedure will be proposed for upper bound limit analysis. We shall employ a radial point interpolation method (RPIM) originally proposed by Wang and Liu [67] to construct the kinematically admissible velocity field. With the built-in property of Kronecker delta function in the shape function of RPIM, it is expected to resolve the first issue concerning the enforcement of boundary conditions. Meanwhile, we shall employ a novel meshless integration technique based on the Cartesian Transformation Method (CTM) developed by Khosravifard and Hematiyan [31]. By using this technique, a domain integration can be sequentially first transformed into a boundary integration and then a one-dimensional (1D) integration, such that no integral background cells are required. A direct iterative method will be used to solve the NLP upper bound problem.

2. Numerical formulation of upper bound approach based on RPIM

2.1. Mathematical description of the upper bound theorem

Under the assumption of small deformation, consider a rigid-perfectly plastic solid V with a boundary S subjected to body forces $\mathbf{g}$ and tractions $\mathbf{t}$ at part of the surface, S_σ. The remaining part of the surface is supposed to be S_u, and $S_\sigma \cup S_u = S$, $S_\sigma \cap S_u = \emptyset$. The upper bound theorem states that the solid will collapse if there exists a kinematically admissible velocity field $\dot{\mathbf{u}} \in U$, such that the rate of external forces work equals to the rate of internal power dissipation. Mathematically, the collapse load multiplier λ can be determined by the following optimization problem:

$$\lambda = \min_{\mathbf{u} \in U} \int_V D(\dot{\boldsymbol{\varepsilon}})dV$$
$$\text{s.t.} \quad W_{\text{ext}}(\dot{\mathbf{u}}) = \int_V \mathbf{g}^T \dot{\mathbf{u}} dV + \int_{S_\sigma} \mathbf{t}^T \dot{\mathbf{u}} \; dS = 1 \tag{1}$$

where $\dot{\boldsymbol{\varepsilon}} = \nabla \dot{\mathbf{u}}$ is the plastic admissible strain rate with ∇ being the linear differential operator. U is a set of kinematically admissible velocity field defined below:

$$U = \{\dot{\mathbf{u}} = \dot{\mathbf{u}}_0 \text{ on } S_u, \; W_{ext}(\dot{\mathbf{u}}) > 0\} \tag{2}$$

In addition, associated flow rule is assumed such that the plastic admissible strain rates can be expressed as follows:

$$\dot{\boldsymbol{\varepsilon}} = \dot{\mu} \frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{3}$$

where $\dot{\mu}$ denotes a non-negative plastic multiplier. In Eq. (1), $D(\dot{\boldsymbol{\varepsilon}})$ denotes the plastic dissipation function which may be defined as follows:

$$D(\dot{\boldsymbol{\varepsilon}}) = \max_{\boldsymbol{\sigma} \in K} \{\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}\} \equiv \boldsymbol{\sigma}_\varepsilon : \dot{\boldsymbol{\varepsilon}} \tag{4}$$

where $\boldsymbol{\sigma}$ is the admissible stresses constrained by the convex yield surface, and $\boldsymbol{\sigma}_\varepsilon$ is the stresses on the yield surface associated with any strain rates $\boldsymbol{\varepsilon}$ through the associated flow rule. K denotes a set of plastic admissible stresses which can be expressed as follows [53]:

$$K = \{\boldsymbol{\sigma} : f\{\boldsymbol{\sigma}\} \le 0\} \tag{5}$$

Evidently, the mathematical optimization problem in (1) is solvable only if a yield function is appropriately specified. The dissipation function in Eq. (4) can be reformulated as follows [7]:

$$D = \sqrt{\dot{\boldsymbol{\varepsilon}}^T \boldsymbol{\Theta} \dot{\boldsymbol{\varepsilon}}} \tag{6}$$

For a plane strain problem, the stress matrix $\boldsymbol{\Theta}$ can be expressed as

$$\boldsymbol{\Theta} = \begin{bmatrix} \sigma_s^2 & -\sigma_s^2 & 0 \\ -\sigma_s^2 & \sigma_s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

whilst for plane stress problem, it can be expressed as

$$\boldsymbol{\Theta} = \frac{1}{3} \begin{bmatrix} 4\sigma_s^2 & -2\sigma_s^2 & 0 \\ -2\sigma_s^2 & 4\sigma_s^2 & 0 \\ 0 & 0 & \sigma_s^2 \end{bmatrix} \tag{8}$$

In both Eqs. (7) and (8) σ_s is the yield stress. Consequently, limit analysis by the upper bound theorem can be recast into the following generalized nonlinear optimization problem:

$$\lambda = \min_{\mathbf{u} \in U} \int_V [\sqrt{\dot{\boldsymbol{\varepsilon}}^T \boldsymbol{\Theta} \dot{\boldsymbol{\varepsilon}}}] dV$$
$$\text{s.t.} \quad \int_V \mathbf{f}^T \dot{\mathbf{u}} dV + \int_{S_\sigma} \mathbf{g}^T \dot{\mathbf{u}} dS = 1 \tag{9}$$

For a practical problem with finite domain, the above mathematical optimization problem may be solved by discretization techniques with such numerical methods as finite element method or mesh-free method. In the following subsection, a NLP scheme in conjunction with mesh-free method will be developed for this purpose.

2.2. Nonlinear programming based on radial point interpolation method

2.2.1. Radial point interpolation method

A radial point interpolation method (RPIM) originally proposed by Wang and Liu [67] will be employed to construct a displacement field for the mesh-free method. RPIM is based on local supporting nodes and includes polynomial reproduction in the