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Limit analysis of frictional block assemblies by means of fictitious associative-type contact interface laws

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ABSTRACT

A new method for the limit analysis of discrete systems formed by dry rigid blocks with Coulomb-type (non-associative) contact interface laws is here exposed. The method resorts to a fictitious system whose cohesive-type contact interface laws depend on the axial forces of the real block system. Two theorems establish the connection between the collapse state of the frictional block assembly and that of the fictitious one. Based on this result, an original problem of mathematical programming is formulated to determine the minimum collapse load multiplier for block assemblies interacting through frictional interfaces. In the proposed formulation the complementarity condition is not introduced as constraint but is obtained as Karush–Khun–Tucker condition. Numerical applications demonstrate the potential of this approach for assessing the collapse load of masonry-like structures.

1. Introduction

The limit analysis provides a simple tool for calculating the collapse load of the structures. Its most appealing feature is the determination, in a single step, of the upper and/or lower bounds at which the collapse occurs. Within this framework, the limit analysis of block assemblies has received a lot of attentions because of its relevant engineering applications. In this sense, the collapse load estimation of rigid block systems interacting through frictional interfaces is of particular importance because of its implications in the assessment, for instance, of ancient masonry structures.

In this field, Livesley [1] proposed a procedure where the load factor is maximized, subjected to the equilibrium equations of the structure and linear constraints imposed by criteria of failure at the block interfaces. On assuming a Coulomb friction at the interface, the method substantially applied the lower bound theorem of the limit analysis but gave an overestimate of the true collapse load, along with an incorrect prediction of the failure mode. Livesley also illustrated a post-optimality analysis for testing the validity of the obtained solution, but no remedy was proposed to deal with the load factor overestimation. Boothby [2] used the lower and upper bound theorems in order to study masonry piers and arches in which sliding between blocks at the joints is allowed. In that study, the kinematic relationships for a general two-dimensional system of rigid blocks were developed. Subsequently, the linear terms were retained to achieve at a linear programming problem from which lower and upper bound theorems were inferred. The limit analysis of bodies in unilateral contact with dry friction was addressed in Ref. [3] by exploiting the so-called implicit standard materials, a class of materials for which the flow rule normality is recovered. Baggio and Trovalusci [4] addressed the limit analysis for no-tension and frictional three-dimensional discrete systems. In their study, the solution of the problem of nonlinear programming is obtained by solving a preliminary problem of linear programming which corresponds to a linearized limit analysis with dilatancy at the interfaces. Ferris and Tin-Loi [5] calculated the collapse loads of discrete rigid block systems with frictional contact interfaces by formulating a special constrained optimization problem, and proposed an algorithm based on the relaxation of the complementarity constraint to solve it. The relaxation parameter is progressively reduced to zero through a succession of nonlinear subproblems. Orduña and Lourenço [6] presented a model for the limit analysis of three-dimensional block assemblages interacting through friction interfaces and included a proposal for torsional failure modes. The model also accounted for limited compressive stresses at the interfaces. Gilbert and co-workers [7] illustrated an iterative procedure based on the successive solution of linear programming sub-problems. The method presented in Ref. [7] assumes fictitious values of cohesion and negative angles of friction which are progressively relaxed toward zero. Recently, a finite element

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based approach was described in Ref. [8] for the limit analysis of planar systems formed from linear elastic bodies in non-penetrative contact with Coulomb friction.

This paper is concerned with the limit analysis of discrete systems formed by dry rigid blocks with a Coulomb-type (non-associative) contact interface laws. The proposed method resorts to a discrete system with fictitious cohesive-type contact interface laws depending on the axial forces of the real block system. Once the connection between the collapse state of the fictitious block assembly and that of the real one is demonstrated, a new formulation of the corresponding mathematical programming problem is proposed. In particular, the minimum collapse multiplier is here obtained by solving a nonlinear mathematical programming problem where the constraints include: (i) equilibrium conditions, (ii) kinematic conditions, and (iii) a further condition which imposes that the collapse multiplier is kinematically admissible for the fictitious system with cohesive-type contact laws. In doing so, the classical complementarity condition is not introduced as constraint but is obtained as Karush–Khun–Tucker condition. The potential of this approach is demonstrated by assessing the collapse load in some masonry-like structures.

2. Limit analysis of frictional block assemblies

2.1. Equilibrium conditions

An assembly of dry n_b blocks is considered. The constituent blocks are rigid and are allowed to slide over each other. Moreover, the Coulomb model is assumed to represent the frictional contact interfaces between the blocks. Contact forces and moments for the jth constituent block of some typical assemblies (i.e., masonry panels and arches) are shown in Fig. 1.

The equilibrium equations for the blocks are

$$\begin{cases} \mathbf{f}_0^j + \alpha\overline{\mathbf{f}}^j + \sum\limits_{\mathbf{x}_{ck} \in \partial\mathcal{B}^j} (V_k \mathbf{t}_k + N_k \mathbf{n}_k) = \mathbf{0} \\ \mathbf{x}_G^j \wedge (\mathbf{f}_0^j + \alpha\overline{\mathbf{f}}^j) + \sum\limits_{\mathbf{x}_{ck} \in \partial\mathcal{B}^j} [(\mathbf{x}_{ck} - \mathbf{x}_G^j) \wedge (V_k \mathbf{t}_k + N_k \mathbf{n}_k) \pm M_k \mathbf{k}] = \mathbf{0} \end{cases} \quad (1)$$

where $j = 1, \ldots, n_b$, $\mathbf{f}_0^j$ denotes the constant external forces acting on the jth block, $\alpha\overline{\mathbf{f}}^j$ denotes the variable external forces acting on the jth block (α is the load multiplier that amplifies the known vector $\overline{\mathbf{f}}^j$). Moreover, N_k are the normal contact forces, V_k are the shear contact forces, M_k are the contact bending moments ($\mathbf{n}_k$, $\mathbf{t}_k$ and $\mathbf{k}$ are unit vectors). The kth resultant internal force or moment acts on $\mathbf{x}_{ck}$, which is the kth contact interface point (with $k = 1, \ldots, n_c$) belonging to the boundary $\partial\mathcal{B}^j$ of the jth block whose center of mass is denoted as $\mathbf{x}_G^j$.

Based on Eq. (1), the equilibrium of the structure can be expressed as follows [5]:

$$\mathbf{A}_f \mathbf{s}_f + \alpha\overline{\mathbf{f}} + \mathbf{f}_0 = \mathbf{A}\mathbf{s} + \mathbf{f}_0 = \mathbf{0} \quad (2)$$

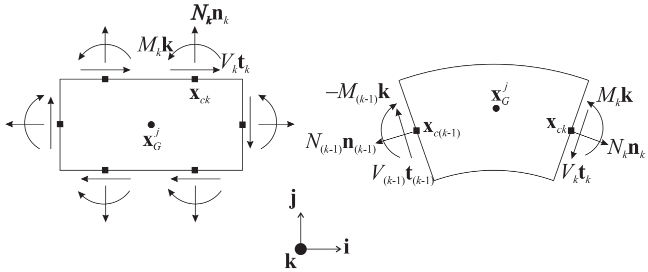

Fig. 1. Contact forces and moments.

in which $\mathbf{s}_f$ is the vector of the contact forces (internal forces and reactions), $\alpha\overline{\mathbf{f}}$ is the vector collecting the variable forces $\alpha\overline{\mathbf{f}}^j$ and $\mathbf{f}_0$ is the vector collecting the constant external forces $\mathbf{f}_0^j$. Moreover, $\mathbf{A} = [\mathbf{A}_f \ \overline{\mathbf{f}}]$. It is also introduced a vector $\mathbf{s}$ which includes the contact forces vector $\mathbf{s}_f$ and the load multiplier α

$$\mathbf{s}^T = \{\mathbf{s}_f^T \ \ \alpha\} = \{\mathbf{s}_N^T \ \ \mathbf{s}_V^T \ \ \mathbf{s}_M^T \ \ \alpha\} = \{N_1 \ldots N_{n_c} \ \ V_1 \ldots V_{n_c} \ \ M_1 \ldots M_{n_c} \ \ \alpha\} \quad (3)$$

For such a frictional system, the following conditions must be met at each contact interface:

$$\begin{cases} \mu N_k + V_k \le 0 \\ \mu N_k - V_k \le 0 \\ d_k N_k - M_k \le 0 \\ d_k N_k + M_k \le 0 \end{cases} \quad (4)$$

where μ is the static friction coefficient and $d_k > 0$ is the maximum eccentricity of the resultant contact force at the kth contact surface. The conditions in Eq. (4) imply that the axial contact forces N_k must be negative or null.

On taking into account all contact interfaces, the conditions in Eq. (4) lead to

$$\mathbf{N}\mathbf{s} = \begin{bmatrix} \mu\mathbf{I} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mu\mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{d} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{d} & \mathbf{0} & -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{s}_N \\ \mathbf{s}_V \\ \mathbf{s}_M \\ \alpha \end{Bmatrix} \le \mathbf{0} \quad (5)$$

in which $\mathbf{d} = \mathrm{diag}\{d_1 \ldots d_k \ldots d_{n_c}\}$, $\mathbf{I}$ is the identity matrix and $\mathbf{0}$ is the null matrix. It is finally denoted with $\mathcal{S}$ the set which collects the statically admissible equilibrium states of the considered block assembly

$$\mathcal{S} = \left\{ \mathbf{s} \,\middle|\, \begin{cases} \mathbf{A}\mathbf{s} + \mathbf{f}_0 = \mathbf{0} \\ \mathbf{N}\mathbf{s} \le \mathbf{0} \end{cases} \right\} \quad (6)$$

Since a Coulomb-type friction behavior without dilatancy is assumed, the resulting contact interface laws are non-associative and, therefore, the classical limit analysis is no longer applicable. As a consequence, a novel problem of mathematical programming is proposed in the following to calculate the minimum limit load multiplier.

2.2. Limit analysis through a fictitious block assembly

It is now considered a fictitious block assembly, identical to that presented before, except that cohesive-type contact interface laws are assumed. The cohesive strengths are taken equal to $-\mu\mathbf{s}_N$, thus depending on the axial contact forces of the real block system. The relationship between the collapse state of the real block system and that of the fictitious one is now investigated.

Theorem. *Given any collapse state of the frictional block assembly, the collapse load multiplier α is always equal to the collapse load multiplier $\alpha_{as}(\mathbf{s}_N)$ of the block assembly with fictitious associative-type contact interface laws .*

Proof. The following conditions must be met at each contact interface of the fictitious system:

$$\begin{cases} \tilde{V}_k \le -\mu N_k \\ -\tilde{V}_k \le -\mu N_k \\ d_k \tilde{N}_k + \tilde{M}_k \le 0 \\ d_k \tilde{N}_k - \tilde{M}_k \le 0 \end{cases} \quad (7)$$

where $\tilde{N}_k$, $\tilde{V}_k$ and $\tilde{M}_k$ are the normal contact forces, the shear contact forces and the contact bending moments of this fictitious system, respectively.