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Analytical and numerical solutions for thick beams with thermoelastic damping

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ABSTRACT

In this work, an analytical solution for thermoelastic damping (TED) quality factor in beams based on Timoshenko beam theory has been proposed along the lines of a previous analytical solution obtained by Lifshitz and Roukes. Heat transfer in the axial direction of the beam was neglected while deriving the analytical solution. A numerical approach using the spectral element method (SEM) was implemented for obtaining the TED quality factor in a Timoshenko beam. Heat transfer in both the axial and thickness directions was considered for obtaining the numerical solution. The two-dimensional heat conduction problem was transformed into an one-dimensional problem by using a weighted residual technique.

Quality factors were obtained numerically using eigenfrequency analysis and energy approach. The results from both analytical method and SEM were compared with the analytical models and three-dimensional finite element solutions. The analytical solution based on Timoshenko beam model gave a better result when compared to the analytical model based on the Euler–Bernoulli model. Numerical solutions are in good agreement with both analytical results and three-dimensional finite element results when the aspect ratio (L/h) is high. The numerical results were closer to the three-dimensional solutions as thickness increased. It has been shown that heat transfer in the axial direction cannot be ignored while computing quality factor attributable to TED in thick beams.

1. Introduction

Thermoelastic damping (TED) is an important phenomenon in case of structures which need to be considered in some specialized environments. In large aerospace structures which need high precision designing [1] and in structures of length scale in micrometers like MEMS resonators and oscillators which operate in vacuum or near vacuum, the TED needs to be quantified accurately. In aerospace, aircraft and submarine structures, TED is a favorable phenomenon which may help to suppress undesirable vibrations, flutter, etc. [2]. On the other hand, TED is detrimental in devices like MEMS oscillators and resonators where high quality factors(Q) are desired [3]. The accurate evaluation of TED can also be used for *a priori* estimation of other types of damping like support loss and fluid damping from experimental data. From an engineering design point of view, the estimation of TED of these structural elements at the design phase is important for accurately designing these elements to obtain the desired functionalities.

Since the stress and temperature fields are coupled in a material, the temperature field changes when it is stressed [4]. If the stress field has some inhomogeneities, there is inhomogeneity in the temperature field as well. Fig. 1 shows a snapshot of the distribution of temperature in a beam undergoing flexural vibration. This inhomogeneity in the temperature leads to heat transfer from the hotter region to the colder region. By second law of thermodynamics, this heat transfer leads to the generation of entropy and loss of mechanical energy in the form of heat.

In a seminal work by Zener [5], the thermoelastic damping in a beam was studied and an analytical formula for the TED quality factor was derived. At the resonant frequency, Zener's equation predicts the TED quality factor for a thin beam undergoing transverse vibrations as

$$Q_{TED}^{-1} = \frac{E\alpha^2 T_0}{\rho C} \frac{\omega_n \tau_1}{1+(\omega_n \tau_1)^2} \quad (1)$$

where ρ is the density, E is the Young's modulus, α is the linear coefficient of thermal expansion, T_0 is the equilibrium temperature of the solid, C is the specific heat per unit mass, h is the thickness of the beam, ω_n is the resonant frequency and τ_1 is the relaxation time defined as

$$\tau_1 = \frac{h^2 \rho C}{\Pi^2 k}. \quad (2)$$

Here k is the thermal conductivity.

Another widely used analytical model was proposed by Lifshitz and Roukes [3]. The change in resonant frequency in the presence

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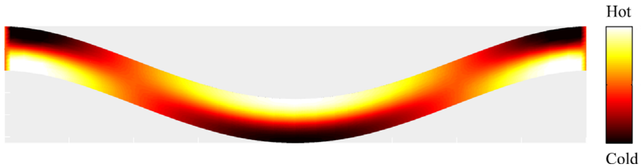

Fig. 1. Temperature distribution in a beam under deformation.

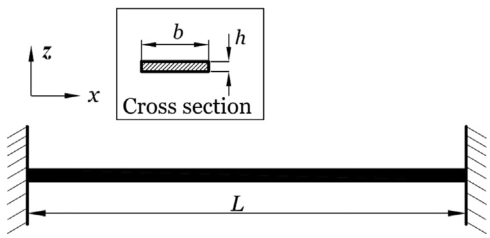

Fig. 2. Schematic of a Timoshenko beam with clamped–clamped boundary conditions.

of TED is taken into account in this model, thereby improving upon Zener's classical model. The equation for the TED quality factor given by Lifshitz and Roukes (LR) is

$$Q_{TED}^{-1} = \frac{E\alpha^2 T_0}{\rho C}\left(\frac{6}{\vartheta^2} - \frac{6}{\vartheta^3}\frac{\sinh\vartheta \sin\vartheta}{\cosh\vartheta \cos\vartheta}\right) \tag{3}$$

where

$$\vartheta = h\sqrt{\frac{\omega_n \rho C}{2k}}. \tag{4}$$

There were similar attempts to quantify the TED in different types of structures. Kinra and Milligan [6] developed a theory to compute the energy lost by measuring the entropy generated in an Euler–Bernoulli beam. De and Aluru [7] modified the Lifshitz and Roukes model to accommodate higher order harmonics for analyzing the electrostatically actuated microbeams. All these analytical models consider very thin beams and account only for heat transfer that occurs through the thickness direction. The thermal boundary conditions at the structural boundaries were also not taken into account in those analyses. There have been two-dimensional theories developed to accommodate these issues. Shieh [8] studied the TED in Timoshenko beam with circular cross section with simply supported boundary conditions. Shieh used the 2-D heat conduction equation coupled with structural domain and obtained the damping factors. Chadwick [9] extended Zener's theory to 3-dimensional analysis of TED and obtained solutions in the form of series expansion of eigenfunctions.

There have been several attempts to solve the TED problems in different domains numerically. Prevost and Tao [10] used finite element method to solve coupled thermoelastic problem. They used second order heat conduction equation instead of the Fourier equation. Prabhakar and Vengallattore [11] coupled 1-D Euler–Bernoulli beam theory and 2-D heat conduction equation and obtained the TED quality factor in terms of series solutions. Serra and Bonaldi [12] presented a finite element formulation for TED in a Reissner–Mindlin plate and 3-D elastic structure. In the plate model, they used a more restrictive linear approximation for temperature variation across the thickness which is justifiable for thin plates but which may not be able to capture the temperature variation of thick plates. Lepage [13] used finite element method to find the TED quality factor of Euler–Bernoulli beam. He used a cubic approximation of temperature variation across the thickness. Guo et al. [14] used two-dimensional finite element method to analyze vented MEMS resonators. De and Aluru [7] used a combined finite cloud method and boundary cloud method to verify the results from their modified theory of TED.

Most of the models existing in the literature use Euler–Bernoulli beam theory which fails at very low length to thickness ratios. The simple analytical models of Zener, Lifshitz and Roukes are easy to implement, but the assumption of negligible heat transfer in the axial direction may be incorrect when the beams are thick. In thick beams, heat conduction in the thickness direction may not be negligible as compared to the heat conduction in the axial direction. In this work, an analytical solution for TED quality factor in 1-D Timoshenko beam coupled with the 1-D thermal conduction equation by neglecting the heat conduction in the axial direction using a procedure similar to that followed by Lifshitz and Roukes [3] is presented to explore the impact of beam theories used to model the structural part on the TED quality factor estimate. In case of numerical solution, the 1-D Timoshenko beam equations coupled with the 2-D thermal conduction equations are used to model the TED. Heat conduction in the axial direction is not neglected in numerical solution. A weighted residual technique is used in the thickness direction to reduce the 2-D heat equation to an equivalent 1-D heat conduction equation. These coupled equations are solved numerically using the spectral element method implemented through MATLAB ® environment. The spectral element method (SEM) is a higher order finite element (FEM) which combines the geometrical flexibility of classical FEM and exponential convergence of global spectral methods [15]. The main advantages of SEM over FEM are high computational efficiency compared to h-type FEM at fixed accuracy level, exponential convergence, optimally lumped mass matrix and efficient tensor product factorization [16].

In the following section, the fully coupled thermoelastic equations for a Timoshenko beam have been formulated to derive an analytical solution for TED quality factor along the lines of Lifshitz and Roukes [3] and to develop a numerical solution using the spectral element method. For obtaining the numerical solution, two-dimensional thermal equation has been reduced to an equivalent one-dimensional equation by applying a weighted residual technique in thickness direction.

2. Thermoelastic vibrations in a linear Timoshenko beam

2.1. Mathematical modelling

Consider a beam of length L $(0 \le x \le L)$, width b $(0 \le y \le b)$ and thickness h $(-h/2 \le z \le h/2)$ as shown in Fig. 2. The displacement field in the Timoshenko beam is given as [17]

$$u_x = u_0 + z\phi(x,t), \tag{5}$$

$$u_y = 0, \tag{6}$$

$$u_z = w_0(x,t) \tag{7}$$

where u_x, u_y and u_z are the displacements in the x, y, z directions respectively. z is the distance of a point from the neutral axis of the beam, u_0 is the axial displacement of a point on the neutral axis, w_0 is the transverse displacement of a point on the neutral axis and ϕ is the rotation of the transverse normal about the $y-$ axis. Using these displacements, the non-zero stress components, σ_{xx} and σ_{xy}, are calculated as

$$\sigma_{xx} = E\left(\frac{\partial u_x}{\partial x} - \alpha\Delta T\right) = E\left(z\frac{\partial\phi}{\partial x} - \alpha\Delta T\right), \tag{8}$$

$$\sigma_{xy} = G\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = G\left(\phi + \frac{\partial w_0}{\partial x}\right), \tag{9}$$