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Exact solutions for an elastic damageable hollow sphere subjected to isotropic mechanical loadings

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ABSTRACT

In this paper, we first recall some available Eshelby-based homogenization schemes applied to microcracked materials. An emphasis is put on models accounting for interacting opened or closed microcracks and their spatial distribution. On the basis of these schemes, we briefly present a class of isotropic damage models. The main part of the study is devoted to the derivation of exact solutions for mechanical fields (damage distribution, displacement, stress fields) in a hollow sphere subjected to a radial loading and made up of an elastic damageable material. The solutions, discussed in link with the different homogenization schemes, may serve as a reference for assessment of numerical predictions of brittle damage models. Interestingly, it is shown that the existence of physically meaningful solutions strongly depends on the model under consideration. Finally, we establish explicit solution to the hollow sphere problem in unloading regime.

1. Introduction

Nonlinear behavior of quasi-brittle materials such as concrete or rocks is mainly attributed to nucleation and growth of microcracks under mechanical loading (see for example [1]). Modeling of the resulting deterioration phenomena can be suitably performed in the framework of Continuum Damage Mechanics (CDM). This can be done by means of a purely macroscopic approach to which an important literature has been devoted (e.g. [2–10], etc.).[1]

A complementary alternative consists in micromechanical studies which aim at deriving the effective behavior and damage of brittle materials based on statistical informations on the microcracks system (e.g. [14–19]). Most of these models issued from the above studies are based on elementary solutions in Linear Elastic Fracture Mechanics (LEFM) and have significantly contributed to the physical understanding of damage mechanisms: in particular, the damage variable to be used at the macroscopic scale is clearly identified (microcracks density parameter). A series of two review papers by Kachanov [20,21] can be notably recommended concerning the LEFM-based micromechanics of microcracked media. Mention has to be made of contributions in this field when the solid matrix displays structural anisotropy (see for instance [22–25], etc.). Other recent developments, in Eshelby-based micromechanics, include studies accounting notably for spatial distribution of microcracks [26–28], or for poromechanical coupling [29,30].

From a structural point of view, the difficulty to deal with damage-induced softening regimes in numerical computation is still well recognized as an crucial question which may still deserve significant research effort. This difficulty remains to be a scientific challenge in so far as it appears as a possible limitation for the transfer of damage models in engineering practice. In this respect, it seems highly desirable to rely upon closed form solutions of simple structural problems, to be used as academical benchmark for numerical solutions. This is the main purpose of the present paper which is organized as follows.

First, various homogenization schemes are described in order to derive the effective stiffness of microcracked materials. A standard thermodynamic reasoning allows us to adapt the concept of energy release rate to the damage evolution problem. In practice, implementation of the damage models involves the derivative of the effective stiffness with respect to the damage variable which is provided by the micromechanical analysis. In view of application to the considered structural problem, only isotropic properties are addressed.

The main part of the study deals with the response of a hollow sphere made up of an elastic damage material, and subjected to traction on the external boundary. The model based on the dilute

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[1] One can also refer to the following textbooks [11,12]. Nonlocal aspects of damage can also be mentioned (see for instance [13]).

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scheme is first implemented and discussed. Thereafter, a more general damage law including the results of both Mori–Tanaka [31] and Ponte-Castaneda and Willis schemes is adopted. The necessary conditions for obtaining a physically meaningful response are identified. When they are satisfied, it is shown that there exists a maximum admissible loading. Beyond this threshold, the softening part of the response is fully described at both the microscopic and macroscopic levels. Unloading phases are also examined.

2. Brief recall of some available estimates for the effective stiffness of microcraked media

We hereafter propose a short review of the basic ideas of the homogenization theory in view of its application to microcracked materials. The interested reader is referred for instance to [32], [29], [30], [33] or [18].

2.1. Basic concepts of homogenization of linear elastic composites

Let us consider a representative element volume Ω, *rev*, subjected, as classically to the uniform strain boundary conditions which consist in prescribing the displacement $\underline{\xi}$ on the boundary[2] $\partial\Omega$:

$$(\forall \underline{z} \in \partial\Omega) \quad \underline{\xi}(\underline{z}) = \mathbf{E} \cdot \underline{z} \tag{1}$$

Eq. (1) mathematically translates the fact that Ω undergoes the strain state represented at the macroscopic scale by the tensor $\mathbf{E}$. For any displacement field $\underline{\xi}$ kinematically admissible with $\mathbf{E}$ in the sense of (1), it is readily seen that the associated strain field $\boldsymbol{\varepsilon}$ meets the "strain average rule":

$$\overline{\boldsymbol{\varepsilon}} = \frac{1}{|\Omega|}\int_\Omega \boldsymbol{\varepsilon}(\underline{z})\, dV_z = \mathbf{E} \tag{2}$$

Let us now illustrate the homogenization procedure in the case of linear elasticity. More precisely, it is assumed that the constitutive equation at point $\underline{z}$ of the microscopic scale takes the linear form $\boldsymbol{\sigma} = \mathbb{C}(\underline{z}) : \boldsymbol{\varepsilon}$, where $\mathbb{C}(\underline{z})$ represents the local stiffness tensor. Note that this framework includes the case of a microcracked linear elastic solid. Indeed, a set of *opened*[3] microcracks $\mathcal{C}_i$ may be represented as the limit case of an linear elastic material with vanishing stiffness. This means that $\mathbb{C}(\underline{z})$ is either equal to the stiffness tensor $\mathbb{C}^s$ of the solid domain[4] Ω^s or to 0 in the cracks:

$$\mathbb{C}(\underline{z}) = \begin{cases} \mathbb{C}^s & (\underline{z} \in \Omega^s) \\ 0 & (\underline{z} \in \mathcal{C}_i) \end{cases} \tag{3}$$

At first sight, the assumption of linear elastic behavior suggests that the response of the *rev* to the uniform strain loading defined by (1) linearly depends on $\mathbf{E}$. This leads to the introduction of the so-called strain concentration tensor $\mathbb{A}(\underline{z})$ relating the macroscopic and microscopic strain tensors:

$$\boldsymbol{\varepsilon}(\underline{z}) = \mathbb{A}(\underline{z}) : \mathbf{E} \tag{4}$$

Introducing the fourth order unit tensor $\mathbb{I}$, the strain average rule classically implies that $\overline{\mathbb{A}} = \mathbb{I}$. The macroscopic state equation then takes the form of a linear relation between the macroscopic stress and strain tensors:

$$\boldsymbol{\Sigma} = \mathbb{C}^{hom} : \mathbf{E} \quad \text{with } \mathbb{C}^{hom} = \overline{\mathbb{C} : \mathbb{A}} \tag{5}$$

$\mathbb{C}^{hom}$ is referred to as homogenized stiffness tensor.

[2] The macroscopic position vector $\underline{x}$ is now omitted in order to simplify the notation.

[3] The case of closed cracks is addressed later.

[4] For simplicity, the solid phase is assumed to be homogeneous.

The above classical result indicates that an estimate of the effective stiffness tensor $\mathbb{C}^{hom}$ is necessarily based on an estimate of the strain (rate) concentration tensor in the cracks. We begin with the so-called dilute homogenization scheme which neglects the mechanical interaction between cracks. We then describe how to capture the latter.

2.2. The dilute scheme

If the mechanical interaction between cracks in the *rev* can be neglected, each crack reacts to the macroscopic loading as if it were embedded in an homogeneous medium having the same stiffness as the solid itself.

2.2.1. Opened microcracks

A simple analytical expression can be derived when microcracks are opened and their orientational distribution is isotropic. Under the dilute concentration assumption, the overall stiffness $\mathbb{C}_{dil}^{hom}$ of the microcracked medium then reads in the form:

$$\mathbb{C}_{dil}^{hom} = \mathbb{C}^s : \left(\mathbb{I} - \frac{4\pi d}{3}\mathbb{Q}\right) \tag{6}$$

where $\mathbb{C}^s$ is the stiffness tensor of the solid matrix embedding the microcracks, while d represents the microcracks density parameter which will be explicitly defined hereafter (see (19)).

The fourth order tensors $\mathbb{J}$ and $\mathbb{K}$ being defined by $J_{ijkl} = \delta_{ij}\delta_{kl}/3$ and $\mathbb{K} = \mathbb{I} - \mathbb{J}$, one has

$$\mathbb{Q} = \frac{4\pi}{3}\langle \mathbb{T} \rangle = Q_1\mathbb{J} + Q_2\mathbb{K} \tag{7}$$

with

$$Q_1 = \frac{16}{9}\frac{1-\nu^{s2}}{1-2\nu^s}; \quad Q_2 = \frac{32}{45}\frac{(1-\nu^s)(5-\nu^s)}{2-\nu^s} \tag{8}$$

It follows that microcrack-induced damage affects the bulk and shear moduli in a different manner:

$$k_{dil}^{hom} = k^s(1 - Q_1 d); \quad \mu_{dil}^{hom} = \mu^s(1 - Q_2 d) \tag{9}$$

This contrasts with the classical isotropic phenomenological damage model in which is assumed

$$k_{dil}^{hom} = k^s(1-d); \quad \mu_{dil}^{hom} = \mu^s(1-d) \tag{10}$$

2.2.2. Closed microcracks

We now consider the case of a closed microcrack. It is assumed that the two faces of the crack are in frictionless contact. This means that the microcrack transmits the normal compressive forces whereas the shear stress in the crack plane remains equal to 0. As opposed to the case of opened microcracks, it is found that the bulk modulus is not affected by the damage. Furthermore, the effect on the shear modulus is different from the one produced by randomly oriented *opened* cracks:

$$k_{dil}^{hom} = k^s; \quad \mu_{dil}^{hom} = \mu^s(1 - Q_2' d) \tag{11}$$

with

$$Q_2' = \frac{32}{15}\frac{1-\nu^s}{2-\nu^s} \tag{12}$$

2.3. Mori–Tanaka estimate [31]

Since the dilute scheme neglects the mechanical interaction between cracks, it is, by nature, restricted to small values of the crack density. The so-called Mori–Tanaka scheme is an attempt to overcome this shortcoming.