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A frictionless contact problem for a flexible circular plate and an incompressible non-homogeneous elastic halfspace

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ABSTRACT

In this paper we apply an energy method to examine the axisymmetric contact problem for a flexible circular plate in smooth contact an incompressible elastic halfspace, where the linear elastic shear modulus varies exponentially with depth. The approach adopted approximates the deflected shape of the plate by a power series expansion which satisfies the kinematics of deformation of the plate and the Kirchhoff boundary condition at the edge of the plate. The coefficients in the series are evaluated by making use of the principle of minimum potential energy. Results are obtained for the maximum deflection, the relative deflection and the maximum flexural moment in the circular plate. The results derived from the proposed procedure are compared with equivalent results derived from a computational procedure.

1. Introduction

The flexural behavior of finite plates resting on the surface of deformable elastic media is of interest to several branches of engineering and in particular to the study of the interaction between foundations and geologic media. The flexural behavior of a loaded circular plate was investigated by Zemochkin [1], Habel [2], and Holmberg [3] who used discretization techniques to represent the contact pressures as a series of concentric annular regions of uniform stress. The classical study by Borowicka [4] examined the influence of the relative rigidity of circular plate, subjected to uniform external load and resting on an isotropic elastic halfspace using a power series expansion technique. Ishkova [5] and Brown [6] presented a modified solution to the problem in which they considered the effect of near edge singular terms in the approximation of the contact stress distribution. An extensive review of various investigations in this area is given by Selvadurai [7,8]. The study by Selvadurai [8] presented the first application of the energy method to examine the elastostatic contact problem where the deflected shape of the circular plate is presented in the form of a power series in terms of the radial coordinate r. Selvadurai [9] also applied the energy method to investigate the behavior of a circular flexible plate embedded in bonded contact with an elastic infinite halfspace. Zaman et al. [10] also used the energy approach advocated in [8,9] to examine the flexural behavior of a uniformly loaded flexible circular plate where the shape of the plate is approximated by an even-order power series expansion in terms of the radial coordinate. Selvadurai et al. [11] have also applied a variational technique to examine the mechanics of a flexible diaphragm in contact with an elastic medium. Pak et al. [12] investigated the tensionless contact of an annular flexible plate with a smooth halfspace under axisymmetric loads. Selvadurai and Dumont [13] utilized the energy method to investigate the contact problem for an isotropic elastic halfspace containing a Mindlin-type axial force and a flexible circular plate subjected simultaneously to an external load $p(r)$.

The variational approach presented by Selvadurai [8] assumes a deflected shape of the plate $w(r)$ in the form of an even-order power series in the radial coordinate. The total potential energy functional is then developed for the plate–elastic halfspace region, which is defined in terms of four undetermined constants characterizing the deflected shape of the plate. Invoking the Kirchhoff boundary conditions applicable to the free edge of the plate we can eliminate two constants in the series and the two remaining constants are evaluated by the minimization of the total potential energy functional of the system.

In this study, the incompressible elastic halfspace is assumed to be non-homogeneous with the shear modulus varying exponentially with depth. The majority of the classical studies have focused on problems where the halfspace region is homogeneous and the indenter is both rigid and axisymmetric e.g. [14–21]. Departures to this model are documented by Gorbunov-Posadov [22], Gladwell [23], Willner [24], Aleynikov [25], Selvadurai [8,9,26–29], Rajapakse

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and Selvadurai [30], Selvadurai and Dumont [13] and Oliveira et al. [31] who examine cases where the contacting body possesses flexural stiffness. A review of investigations related to contact problems relevant to a non-homogeneous elastic halfspace region is given by Selvadurai and Katebi [32].

The result of the investigation indicates the influence of the relative stiffness of the plate and the elastic non-homogeneity on the deflections and flexural moments in the plate. The results of this study have potential applications to the modeling of structural foundations resting on geologic media. Recent years, the indentation problem involving rigid indenters have been used to examine the mechanical behavior of functionally graded materials. The present study adds to the modeling by introducing both the flexibility of the indenter and the non-homogeneity of the elastic medium.

2. Proposed analytical procedure

Referring to Fig. 1, we examine the problem of the axisymmetric indentation of an incompressible ($\nu = 1/2$), non-homogeneous elastic halfspace by a flexible circular plate of thickness h and radius a. The plate is subjected to a uniform load of intensity p_0 over its entire surface. The non-homogeneity considered in the paper assumes that the shear modulus of the elastic medium varies exponentially according to

$$G(r,\ z) = G_0 e^{\tilde{\lambda} z}\ ;\ r \in (0,\ \infty)\ ;\ z \in (0,\ \infty) \tag{1}$$

where G_0 is constants. Non-dimensional parameter for non-homogeneity ($\tilde{\lambda}$) is then defined by $\lambda = \tilde{\lambda}/a$.

It is further assumed that there is no loss of contact at the frictionless interface. Therefore, the interface displacement can be represented as the deflected shape of the plate, which is identical to the surface displacement of the halfspace in the z-direction over the contact region $0 \le r \le a$. The variational technique proposed by Selvadurai [8] assumes the plate deflection $w(r)$, can be specified to within a set of arbitrary constants, in which the deflected shape of the circular plate can be presented in the form of an even-order power series in terms of the radial coordinate r. The form of $w(r)$is also chosen to satisfy kinematic constraints of the axisymmetric plate deflection. The analysis of the interaction problem via the energy method requires the development of the total potential energy functional for the loading and the plate–elastic halfspace region, which consists of (i) the flexural energy of the plate, (ii) the strain energy of the halfspace region, and (iii) the potential energy of the applied loads. The total potential energy functional can be expressed in terms of the undetermined constants characterizing the deflected shape of the plate. Invoking the Kirchhoff boundary conditions applicable to the free edge of the plate we can eliminate two of the constants in the series and the two remaining constants are evaluated through the minimization of the total potential energy functional of the system.

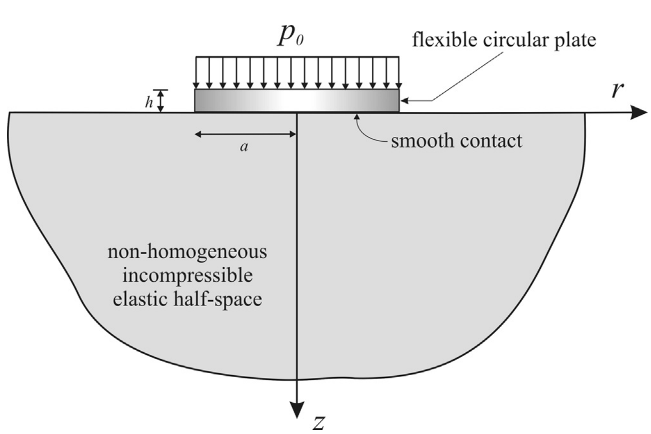

Fig. 1. Uniformly loaded flexible circular plate on the surface of an incompressible elastic halfspace.

3. Variational approach

The proposed formulation is discussed briefly in this section. It is assumed that the deflected shape of the plate can be approximated by the power series expansion:

$$w(r) = a \sum_{i=0}^{3} C_{2i} \left(\frac{r}{a}\right)^{2i} \tag{2}$$

where C_{2i} is the arbitrary constant. The assumed form of the plate deflection (2) has a kinematically admissible form which gives finite rotation and curvature in the plate region. Of the four arbitrary constants, two can be determined by invoking the Kirchhoff boundary conditions applicable to the free edge of the circular plate (Selvadurai [33]), i.e,

$$M_r(a) = -D\left[\frac{d^2 w(r)}{dr^2} + \frac{\nu_b}{r}\frac{dw(r)}{dr}\right]_{r=a} = 0 \tag{3}$$

$$Q_r(a) = -D\left[\frac{d}{dr}\{\nabla^2 w(r)\}\right]_{r=a} = 0 \tag{4}$$

where ν_b is Poisson's ratio of the plate material. The assumed expression for the plate deflection can be reduced to the form [8]

$$w(r) = a\left[C_0 + C_2\left\{\frac{r^2}{a^2} + l_1\frac{r^4}{a^4} + l_2\frac{r^6}{a^6}\right\}\right] \tag{5}$$

where

$$l_1 = \frac{-3(1+\nu_b)}{4(2+\nu_b)};\quad l_2 = \frac{(1+\nu_b)}{6(2+\nu_b)} \tag{6}$$

In the following, the total potential energy functional for the plate–elastic halfspace region is developed using the proposed plate deflection $w(r)$.

3.1. Flexural energy of the plate

The first component of the total energy functional corresponds to the flexural energy of the plate. The flexural behavior of the elastic plate is described by the Poisson–Kirchhoff thin plate theory. This, the flexural energy of the plate with an axisymmetric deflection $w(r)$ is given as follows:

$$U_F = \frac{D}{2}\iint_S \left[\{\nabla^2 w(r)\}^2 - \frac{2(1-\nu_b)}{r}\frac{dw(r)}{dr}\frac{d^2 w(r)}{dr^2}\right] r dr d\theta \tag{7}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr};\quad D = \frac{G_b h^3}{6(1-\nu_b)} \tag{8}$$

and G_b and ν_b are the constant shear modulus and Poisson's ratio of the plate material, respectively, and s represents the plate region.

3.2. Strain energy of the halfspace region

The second component of the total potential energy functional corresponds to the strain energy of the incompressible non-homogeneous elastic halfspace, which is subjected to the displacement field $w(r)$ in the contact region $0 \le r \le a$. The elastic strain energy can be developed by evaluating the work component of the