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Transverse vibration of free–free beams carrying two unequal end masses

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ABSTRACT

Transverse vibration of free–free slender beams carrying two tip masses is studied. An exact frequency equation is derived and the natural frequencies are calculated. An alternative Fredholm integral equation approach is presented to determine resonant frequencies of a vibrating mass–beam system. Simple approximate expressions for fundamental frequencies with high accuracy are obtained. Analytical and approximate results of the frequencies are computed and compared. The effects of concentrated masses at two ends on the natural frequencies and mode shapes are discussed. The results are helpful to design micromechanical sensors and energy harvesters.

1. Introduction

Vibration analysis of a beam carrying additional mass is of significance in engineering applications and a key problem to be solved is to determine the change in natural frequencies due to attached mass. A vibrating beam–mass system can be found both in classical engineering fields and in burgeoning modern micro/nanotechnique fields such as micro/nanoelectromechanical system, AFM probes [1], and nanoscale sensors [2]. Great progress has been made in determining the natural frequencies of a beam–mass system, and a large number of research results have been collected in books [3,4], including frequently encountered cantilevers, clamped beams, and simply supported beams.

Free–free beams are a class of special beams without end restraints. This class of beams has been widely used. For example, the mechanical behavior of flexible missiles and space structures propelled by a rocket thrust can be modeled as free–free beams with a follower load at one end [5–7]. Due to the high-Q performance of free–free resonators superior to that of resonators with other end restraints such as clamped end in measuring very high frequencies (VHF), they are promising candidates for realizing monolithic integration of circuits and resonators, precision filtering and timing, and transceivers for wireless communication [8,9]. Recently, integrated CMOS-MEMS free–free beam resonator arrays are studied to minimize anchor losses and to achieve improved performance [10].

For the applications mentioned above, a fundamental issue is to determine the natural frequencies of a free–free beam. Kirk and Wiedemann studied the natural frequencies and mode shapes of a free–free beam with large end masses [11]. A spinning unconstrained beam with a concentrated mass and subjected to a thrust was analyzed by Yoon and Kim [12]. For a free–free beam with identical concentrated end masses, Haener [13] first solved the problem and gave an approximate expression for the fundamental frequencies. Unfortunately, the approximate fundamental frequencies deviate far from the exact ones, which has been pointed out by Erturk and Inman [14]. Using the curve fitting technique, Erturk and Inman [14] obtained several approximate natural frequencies with high accuracy. However, their results are only suitable for a free–free beam with two identical end masses. For a free–free beam carrying unequal end masses, there is little information on the natural frequencies along with their approximations. On the contrary, for other beams with attached mass, great progress has been made. Some recent researches on resonant frequencies of a beam–mass system can be found, see e.g. [15–18].

This paper aims at solving a free–free beam with two unequal concentrated masses at both tips. The frequency equation is obtained and the natural frequencies are exactly determined. However, the solution of the frequency equation cannot be obtained analytically in closed form due to its transcendental nature, an alternative integral equation approach is further formulated to derive a simple

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approximate expression for the fundamental frequencies. Two closed-form expressions for the fundamental frequencies are given and compared with the exact ones. Mode shapes of free vibration are plotted and discussed. Pinned–pinned and pinned–free beams are recovered from the present with specified tip masses.

2. Exact frequency equation

Consider transverse free vibration of a slender free–free beam carrying two concentrated masses at both tips, as shown in Fig. 1. We denote the deflection as w and the beam length as L. According to Euler–Bernoulli theory of beams [19], the governing equation reads

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}=0, \quad 0<x<L \tag{1}$$

where EI is the bending stiffness of the beam, ρA the distributed mass of the beam, A and I are the cross-sectional area and its second moment, respectively. Here x is the axial coordinate, and t is the time.

Two unequal concentrated masses m_1 and m_2 are attached to both ends $x=0, L$, respectively. Thus one can get the following boundary conditions:

$$EI\frac{\partial^2 w}{\partial x^2}=0, \quad EI\frac{\partial^3 w}{\partial x^3}=-m_1\frac{\partial^2 w}{\partial t^2}, \quad x=0, \tag{2}$$

$$EI\frac{\partial^2 w}{\partial x^2}=0, \quad EI\frac{\partial^3 w}{\partial x^3}=m_2\frac{\partial^2 w}{\partial t^2}, \quad x=L. \tag{3}$$

For harmonic vibration, we take $w=\tilde{w}(x)e^{i\omega t}$, and Eq. (1) becomes

$$EI\frac{\partial^4 \tilde{w}}{\partial x^4}-\rho A\omega^2\tilde{w}=0, \quad 0<x<L. \tag{4}$$

For convenience, the following nondimensional variables are defined:

$$\tilde{w}(x)=W(\xi), \quad \Omega=L^2\omega\sqrt{\frac{\rho A}{EI}}, \quad \xi=\frac{x}{L}, \quad \beta_1=\frac{m_1}{\rho AL}, \quad \beta_2=\frac{m_2}{\rho AL} \tag{5}$$

Under such circumstances, Eq. (4) is rewritten as

$$W^{IV}-\Omega^2 W=0, \quad 0<\xi<1, \tag{6}$$

subjected to the boundary conditions

$$W''(0)=0, \quad W'''(0)=\beta_1\Omega^2 W(0) \tag{7}$$

$$W''(1)=0, \quad W'''(1)=-\beta_2\Omega^2 W(1) \tag{8}$$

where the prime denotes differentiation with respect to the argument.

Using a standard method, we solve the ordinary differential equation (6) and its general solution can be readily obtained to be

$$W=C_1\cos(\lambda\xi)+C_2\sin(\lambda\xi)+C_3\cosh(\lambda\xi)+C_4\sinh(\lambda\xi) \tag{9}$$

where the frequency parameter $\lambda=\sqrt{\Omega}$.

After applying the solution (9) to the boundary conditions (7) and (8), we obtain

$$-C_1+C_3=0 \tag{10}$$

$$-C_2+C_4=\beta_1\lambda(C_1+C_3) \tag{11}$$

$$-C_1\cos\lambda-C_2\sin\lambda+C_3\cosh\lambda+C_4\sinh\lambda=0 \tag{12}$$

$$\begin{aligned}C_1\sin\lambda-C_2\cos\lambda+C_3\sinh\lambda+C_4\cosh\lambda \\ =-\beta_2\lambda[C_1\cos\lambda+C_2\sin\lambda+C_3\cosh\lambda+C_4\sinh\lambda]\end{aligned} \tag{13}$$

Since a system of algebraic equations (10)–(13) has a nontrivial solution, the determinant of the coefficient matrix has to vanish,

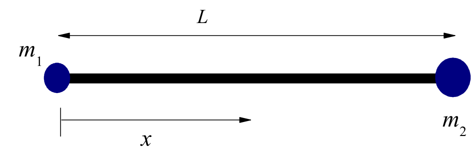

Fig. 1. Schematic of a free–free beam carrying two tip masses.

which leads to the desired frequency equation:

$$1-\cos\lambda\cosh\lambda+\lambda(\beta_1+\beta_2)(\sin\lambda\cosh\lambda-\cos\lambda\sinh\lambda)+2\lambda^2\beta_1\beta_2\sin\lambda\sinh\lambda=0. \tag{14}$$

This is the exact frequency equation of transverse vibration of a free–free beam carrying two tip masses.

In the following, we consider several special cases:

(1) Two identical tip masses are attached: For a free–free beam carrying two equal concentrated masses at each tip, we have $m_1=m_2=m$ (say), i.e. $\beta_1=\beta_2=\beta$ (say). Thus, the above-derived frequency equation (14) reduces to

$$1-\cos\lambda\cosh\lambda+2\lambda\beta(\sin\lambda\cosh\lambda-\cos\lambda\sinh\lambda)+2\lambda^2\beta^2\sin\lambda\sinh\lambda=0, \tag{15}$$

which is in exact agreement with that obtained by Erturk and Inman [14].

(2) One tip mass is attached: For a free–free beam carrying one concentrated mass at a tip, we suppose $m_1\neq 0, m_2=0$, i.e. $\beta_2=0$. Thus, the above-derived frequency equation (14) reduces to

$$1-\cos\lambda\cosh\lambda+\lambda\beta_1(\sin\lambda\cosh\lambda-\cos\lambda\sinh\lambda)=0. \tag{16}$$

(3) No any additional mass is attached: For a free–free beam without any additional mass, the well-known frequency equation

$$1-\cos\lambda\cosh\lambda=0 \tag{17}$$

is recovered from (14) with $\beta_1=\beta_2=0$.

3. Approximate fundamental frequencies

In the foregoing section, an exact frequency equation has been obtained, and so the natural frequencies are easily determined by solving the frequency equation. Nevertheless, Eq. (14) is a transcendental equation, and hence its solution cannot be given in a closed form or in an explicit expression. This is inconvenient especially for engineering applications. To get closed-form natural frequencies, in what follows an alternative Fredholm integral equation approach is presented. To this end, we set

$$\kappa(\xi)=\frac{d^2W}{d\xi^2}, \tag{18}$$

corresponding to the curvature of the amplitude curve W. Thus, W can be expressed by

$$W=\int_0^{\xi}(\xi-s)\kappa(s)\,ds+C\xi+D \tag{19}$$

where C, D are two constants to be determined. Substituting (18) and (19) into the governing equation (6) yields

$$\kappa''-\Omega^2\left[\int_0^{\xi}(\xi-s)\kappa(s)\,ds+C\xi+D\right]=0 \tag{20}$$

After integrating both sides of Eq. (20) twice with respect to ξ, we get

$$\kappa-\frac{\Omega^2}{2}\int_0^{\xi}(\xi-s)^2\kappa(s)\,ds=\frac{1}{2}C\Omega^2\xi^2+D\Omega^2\xi+E\Omega^2 \tag{21}$$