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# Fundamental frequency of the laminated composite cylindrical shell with clamped edges

A.V. Lopatin [a], E.V. Morozov [b,*]

[a] Department of Aerospace Engineering, Siberian State Aerospace University, Krasnoyarsk, Russia
[b] School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra, Australia

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ABSTRACT

Free vibrations of the laminated composite cylindrical shell with clamped edges are considered in this paper. Equations of the theory of laminated shells taking into account the average transverse shear strains are employed for the vibration analysis. A solution of the equations of motion of the shell is based on the Fourier decomposition and the Galerkin method and yields an analytical formula for the calculation of a fundamental frequency. Applications of this formula to the determination of the fundamental frequencies for the filament-wound composite cylindrical shells are demonstrated using numerical examples. The calculations have been verified by comparison with a finite-element solution. It has been shown that the analytical formula presented in this paper provides an efficient means for rapid and reliable calculation of the fundamental frequency which can be used for the assessment of the structural stiffness of the shells in the design analysis.



## 1. Introduction

Laminated composite cylindrical shells manufactured by filament-winding are widely used in various structural applications in aerospace, civil, marine, oil and gas, and other industries. Often the dynamic analysis is required as part of the design procedure for such shells. Conventionally, a finite-element analysis is employed to determine vibration modes and frequencies. However, in many design and analysis cases there is a need to determine a fundamental frequency of the structure avoiding the use of numerical modelling and computational technique. One of the reasons is that the fundamental frequency provides a convenient criterion of the mass and stiffness assessment of the shell design. This is due to the fact that this parameter reflects a combined mutual effect of the bending stiffness and the mass-per-unit-length of the shell wall. Thus, it is desirable to have a compact analytical formula for rapid calculations of the fundamental frequency with sufficient accuracy when designing composite structural components.

The composite cylindrical shells with both ends clamped are commonly used in structural applications. Hence there is a substantial practical interest to the design analysis methods providing the way of determining fundamental frequency of the shells with this type of support. The corresponding vibration problems for the clamped–clamped cylindrical shells attracted attention of many researchers over the years. A number of solutions obtained for isotropic, structurally-anisotropic, and laminated composite cylindrical shells with various types of support can be found in the handbook published by Leissa [1] and in articles [2–9].

Semi-analytical approaches to the free vibration analyses of axisymmetric laminated shells with various combinations of boundary conditions were developed by Pinto Correia et al. [10,11] and Santos et al. [12]. Liu et al. derived exact characteristic equations for free vibrations of thin-walled orthotropic cylindrical shells [13]. A few interesting studies of the dynamic behaviour of the clamped–clamped cylindrical shells were reported in the literature. In the article [14] published by Zhang, the wave propagation approach was extended to analyse the natural frequencies of a cross-ply laminated composite cylindrical shell with the clamped–clamped ends. The natural modes of the shell vibration were considered as a combination of standing waves in the circumferential and axial directions. Hufenbach et al. [15] studied the vibration and damping behaviour of multi-layered composite cylindrical shell. They used the beam functions to satisfy the boundary conditions at the clamped edges of the shell. A numerical analysis of circular cylindrical shells by the local adaptive differential quadrature method (LaDQM), which employed both localised interpolating basis functions and exterior grid points for clamped–clamped boundary treatments, was performed by Zhang et al. [16]. In their study, the motion equations were derived based on the Goldenveizer–Novozhilov shell theory. In the article published by Dai et al. [17], an exact series solution for the vibration analysis of

* Corresponding author.
*E-mail address:* e.morozov@adfa.edu.au (E.V. Morozov).

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circular cylindrical shells with arbitrary boundary conditions was obtained, using the equations based on Flügge's theory. Each of the three displacements was represented by a Fourier series and auxiliary functions and sought in a strong form by letting the solution exactly satisfied both the governing differential equations and the boundary conditions on a point-wise basis. An energy-oriented modified Fourier method has been employed to analyse vibrations of isotropic and composite closed and open cylindrical shells with various boundary conditions by Jin et al. [18,19] and Ye et al. [20].

In this paper, a solution of the free vibration problem is presented for the laminated composite cylindrical shell with the clamped edges. The motion of the shell was modelled by the equations of the laminated shell theory that take into account the transverse shear strains. The governing system of differential equations, determining the shape of the vibrating shell, is of the tenth order for both circumferential and longitudinal coordinates. The problem has been solved using Fourier decomposition and the Galerkin method. As a result, an analytical formula for the calculation of the fundamental frequency of the shells under consideration has been obtained. The formula has been verified by comparison with a finite-element solution and its efficiency is demonstrated for design analysis of the filament-wound composite shell for which the fundamental frequency is adopted as the stiffness criterion.

## 2. Governing equations and solution procedure

Consider a laminated composite cylindrical shell of length $l$ shown in Fig. 1. Its layered wall structure is symmetrical with respect to the middle surface of the shell with the radius $R$. The middle surface is referred to the curvilinear coordinate frame $\alpha, \beta$, and $\gamma$ (see Fig. 1). The coordinate $\alpha$ is parallel to the axis of the cylinder, $\beta$ is the hoop coordinate, and $\gamma$ is normal to the middle surface. The ends of the shell $\alpha=0$ and $\alpha=l$ are fully clamped.

It is assumed that the shell is subjected to the time dependent loading. In this case, the shell motion can be described by the following equations [21]:

$$\frac{\partial N_\alpha}{\partial \alpha}+\frac{\partial N_{\alpha\beta}}{\partial \beta}-B_\rho\frac{\partial^2 u}{\partial \tau^2}=0$$
$$\frac{\partial N_{\alpha\beta}}{\partial \alpha}+\frac{\partial N_\beta}{\partial \beta}+\frac{Q_\beta}{R}-B_\rho\frac{\partial^2 v}{\partial \tau^2}=0$$
$$\frac{\partial Q_\alpha}{\partial \alpha}+\frac{\partial Q_\beta}{\partial \beta}-\frac{N_\beta}{R}-B_\rho\frac{\partial^2 w}{\partial \tau^2}=0$$
$$\frac{\partial M_\alpha}{\partial \alpha}+\frac{\partial M_{\alpha\beta}}{\partial \beta}-Q_\alpha-D_\rho\frac{\partial^2 \theta_\alpha}{\partial \tau^2}=0$$
$$\frac{\partial M_{\alpha\beta}}{\partial \alpha}+\frac{\partial M_\beta}{\partial \beta}-Q_\beta-D_\rho\frac{\partial^2 \theta_\beta}{\partial \tau^2}=0 \quad (1)$$

where $N_\alpha, N_\beta$ and $N_{\alpha\beta}$ are the membrane stress resultants, $M_\alpha, M_\beta$ and $M_{\alpha\beta}$ are the bending and twisting moments (resultant couples), and $Q_\alpha$ and $Q_\beta$ are the shear forces. The terms containing the second derivatives of the displacements $u$, $v$ and deflection $w$ with regard to time $\tau$ correspond to the respective motions of the points of middle surface in the directions of axes $\alpha$, $\beta$ and $\gamma$. The terms including the second derivatives of the angles of rotation $\theta_\alpha$ and $\theta_\beta$ with regard to time correspond to the rotations of the element normal to the middle surface in the planes $\alpha\gamma$ and $\beta\gamma$, respectively. Inertia properties of the laminated wall of the shell, in-plane and under bending, are characterised by the coefficients $B_\rho$ and $D_\rho$, respectively.

The equations of motion, Eq. (1), should be supplemented with the constitutive equations, strain–displacement relationships, and boundary conditions. For the symmetrical structure of the laminated wall, the constitutive equations are given by

$$N_\alpha=B_{11}\varepsilon_\alpha+B_{12}\varepsilon_\beta, N_\beta=B_{21}\varepsilon_\alpha+B_{22}\varepsilon_\beta$$
$$N_{\alpha\beta}=B_{33}\varepsilon_{\alpha\beta}$$
$$M_\alpha=D_{11}\kappa_\alpha+D_{12}\kappa_\beta, M_\beta=D_{21}\kappa_\alpha+D_{22}\kappa_\beta$$
$$M_{\alpha\beta}=D_{33}\kappa_{\alpha\beta}$$
$$Q_\alpha=S_\alpha\psi_\alpha, Q_\beta=S_\beta\psi_\beta \quad (2)$$

where $\varepsilon_\alpha, \varepsilon_\beta$, and $\varepsilon_{\alpha\beta}$ are the membrane strains characterising tension, compression and shear of the shell middle surface, $\kappa_\alpha, \kappa_\beta$ and $\kappa_{\alpha\beta}$ determine bending and twisting of the middle surface, and $\psi_\alpha$ and $\psi_\beta$ are transverse shear strains of the wall. The coefficients $B_{11}, B_{12}, B_{22}, B_{33} (B_{21}=B_{12})$; $D_{11}, D_{12}, D_{22}, D_{33}$ $(D_{21}=D_{12})$; and $S_\alpha, S_\beta$ are the membrane, bending, and transverse shear stiffnesses of the wall, respectively.

The strain–displacements relationships linking the strains $\varepsilon_\alpha$, $\varepsilon_\beta$, $\varepsilon_{\alpha\beta}$, $\kappa_\alpha$, $\kappa_\beta$, $\kappa_{\alpha\beta}$, $\psi_\alpha$, and $\psi_\beta$ with the displacements $u$, $v$, deflection $w$, and angles of rotation $\theta_\alpha$ and $\theta_\beta$ have the following form:

$$\varepsilon_\alpha=\frac{\partial u}{\partial \alpha}, \quad \varepsilon_\beta=\frac{\partial v}{\partial \beta}+\frac{w}{R}, \quad \varepsilon_{\alpha\beta}=\frac{\partial u}{\partial \beta}+\frac{\partial v}{\partial \alpha}$$
$$\kappa_\alpha=\frac{\partial \theta_\alpha}{\partial \alpha}, \quad \kappa_\beta=\frac{\partial \theta_\beta}{\partial \beta}, \quad \kappa_{\alpha\beta}=\frac{\partial \theta_\alpha}{\partial \beta}+\frac{\partial \theta_\beta}{\partial \alpha}$$
$$\psi_\alpha=\theta_\alpha+\frac{\partial w}{\partial \alpha}, \quad \psi_\beta=\theta_\beta-\frac{v}{R}+\frac{\partial w}{\partial \beta} \quad (3)$$

The boundary conditions reflecting the fully clamped ends of the shell $\alpha=0$ and $\alpha=l$ are

$$u=0\,, \; v=0, w=0, \theta_\alpha=0, \theta_\beta=0 \quad (4)$$

Substituting Eqs. (2) and (3) into Eq. (1) yields the following governing equations expressed in terms of displacements $u$, $v$, deflection $w$, and angles of rotation $\theta_\alpha$ and $\theta_\beta$:

$$B_{11}\frac{\partial^2 u}{\partial \alpha^2}+B_{33}\frac{\partial^2 u}{\partial \beta^2}+(B_{12}+B_{33})\frac{\partial^2 v}{\partial \alpha \partial \beta}+\frac{B_{12}}{R}\frac{\partial w}{\partial \alpha}-B_\rho\frac{\partial^2 u}{\partial \tau^2}=0$$

$$(B_{21}+B_{33})\frac{\partial^2 u}{\partial \alpha \partial \beta}+B_{33}\frac{\partial^2 v}{\partial \alpha^2}+B_{22}\frac{\partial^2 v}{\partial \beta^2}$$

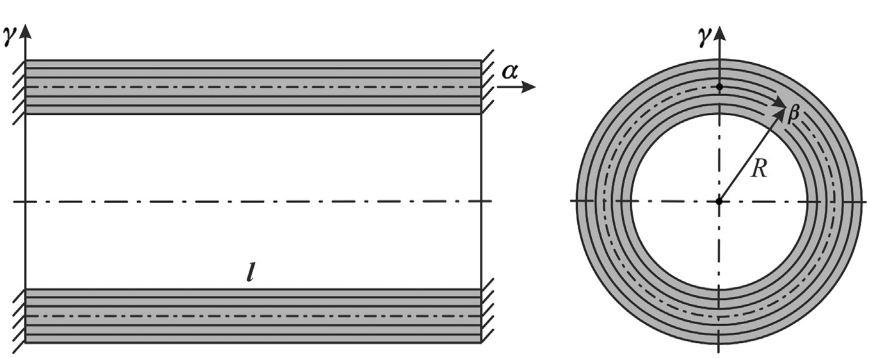


**Fig. 1.** Clamped–clamped laminated composite cylindrical shell.