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Analytical solutions for thermoelastic vibrations of beam resonators with viscous damping in non-Fourier model

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ABSTRACT

Thermoelastic damping is recognized as a significant loss mechanism at room temperature in micro-scale beam resonators in vacuum. In addition, if the resonator operates in air, the viscous damping should be considered. In this study, the forced vibration of beam subjected to a harmonic external force and with the viscous and thermoelatic dampings simultaneously is investigated. Moreover, the heat conduction in the *C–V* model is considered. The analytical solutions of the system with different boundary conditions are presented. Moreover, the mathematical model of the quality factor (Q-factor) of the system is derived. Finally, the effects of the thermal diffusion, the phase lag for heat flux, the squeezing film damping, the size scale and the boundary conditions on the response ratio and the Q-factor of the system are investigated. It is found that the effects of the thermal diffusion, the squeezing film damping, the size scale and the boundary conditions on the response ratio and the Q-factor are significant.

1. Introduction

Micro- and nano-scale resonators have many important technological applications. [1–3]. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high-performance components.

Several different dissipation mechanisms in MEMS have been discussed. These factors includes (1) doping impurities losses, (2) support-related losses, (3) thermoelastic damping, and (4) the radiation of energy away from the resonator into its surroundings [3–10]. It is very difficult to investigate directly the coupled system with several dissipation mechanisms simultaneously. In most literature, in neglect of the coupled effect each independent quality factor for some dissipation $\{Q_{TED}, Q_{fluid}, Q_{anchor}, Q_{surface}, Q_{others}\}$ is firstly determined and the quality factor of the coupled system is approximated as [8,10]

$$\frac{1}{Q} = \frac{1}{Q_{TED}} + \frac{1}{Q_{fluid}} + \frac{1}{Q_{anchor}} + \frac{1}{Q_{surface}} + \frac{1}{Q_{others}}$$

In fact, the coupled effect of these dissipations on the performance of the system should be significant. Obviously, the solution of the above equation is rough.

So far, it is found that thermoelastic damping is a significant loss mechanism in MEMS resonators. Several literatures investigated the effect of thermoelastic damping. Zener [5] experimentally verified the thermoelastic damping process. Boley [11] analyzed the thermally induced vibrations of a simply supported rectangular beam. Manolis and Beskos [12] studied the effect of damping on the vibration of beams subjected to fast surface heating. However, the coupling between stress and temperature fields was not considered. Roszhardt [13] observed thermoelastic damping in silicon micro-resonators at room temperature. Givoli and Rand [14] studied the effect of thermoelastic damping on dynamic response properties of a rod. They found that when the thermal frequency approaches to the critical frequency of rod, the dynamic response is changed significantly. Harrington et al. [15] measured mechanical dissipation in micron-sized single-crystal resonators in torsion and flexural modes. They found that the resonance frequency changes with temperature. Lifshitz and Roukes [16] and Guo and Rogerson [17] investigated thermoelastic damping of a beam with rectangular cross-sections based on the classical Fourier thermal conducting equation. Houston et al. [18] studied the importance of thermoelastic damping for silicon-based MEMS. Their results indicate that the internal friction arising from this mechanism is significant. Sun et al. [19] investigated thermoelastic damping in micro-beam resonators. The thermoelastic damping of micro-beam resonators is analyzed by using both the finite sine Fourier transformation method combined with Laplace transformation and the normal mode analysis. When the thickness of the micro-beam is larger than its characteristic size, the effect of thermoelastic damping weakens as the beam thickness increases. The size-effect induced by thermoelastic coupling would disappear when the thickness of the micro-beam is over a critical value that depends on the material

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properties and the boundary conditions. Sun et al. [20] investigated the laser-induced vibrations of micro-beams under different boundary conditions without the effect of fluid damping. An analytical–numerical technique based on the Laplace transform is used to calculate the vibration of the deflection and thermal moment. The damping ratio and resonant frequency shift ratio of beams due to the air damping effect and the thermoelastic damping effect are compared. Choi et al. [9] presented some efficient prediction of the quality factors of micromechanical resonators. A finite element formulation based on the weak form of fully coupled thermoelastic problems was suggested. Further, the authors applied the model order reduction (MOR) scheme to the coupled multiphysical problem in order to achieve computational efficiency. Vahdat and Rezazadeh [2] investigated the effects of axial and residual stresses on thermoelastic damping in micro-beam resonators. A Galerkin based finite element formulation has been used to analyze TED for the first mode of vibration of the micro-beam resonator with both ends clamped and isothermal. It was found that compressive (tensile) residual stresses increase (decrease) the TED ratio considerably.

Obviously, if a resonator is in absolute vacuum, the fluid damping is zero. However, when a resonator operates in air, the fluid damping is significant. Further, if a beam resonator operating in air is close to the boundary, the squeezing film damping should be considered. The squeezing pressure force between the resonator's beam and the boundary surface is inversely proportional to d_s^3 where d_s is the distance between the beam and the boundary surface [21,22]. Zhang et al. [7] studied the effect of air damping on the frequency response and the quality factor of a micro-machined beam resonator. Their results indicate that air damping generally shifts the resonance frequency on the order of no more than 10^{-6} and degrades the quality factor, and that this effect of air damping increases as the dimension of the beam decreases. However, the effect of squeezing film damping was not considered. In addition to beam resonator, a few literatures are dedicated to the study of thermoelastic damping in micro-plate and tube resonators [3,23,24].

From the above summary, most literatures investigating thermoelastic damping were based on the Fourier conduction model. Moreover, the effect of squeezing film was not taken into consideration. In this paper, the forced vibration model of beam resonator with both effects of the squeezing film and thermoelastic dampings is established based on the *C–V* model. The analytical solution of the system is derived. Moreover, the effects of several parameters on the response ratio, thermal moment and Q-factor are investigated.

2. Governing equation and boundary conditions

Consider the thermoelastic vibration of a Bernoulli–Euler cantilever subjected to the harmonic exciting force. The heat conduction is in the *C–V* model. The isothermal—isolated boundary conditions are firstly investigated. The different boundary conditions are listed in Appendix A–C. The coupled dimensionless governing equations are

$$\frac{\partial^4 w}{\partial \xi^4}+A_1\frac{\partial^2 w}{\partial \tau^2}+c\frac{\partial w}{\partial \tau}+\frac{\partial^2 \theta_T}{\partial \xi^2}=f(\xi)\sin\,\omega\tau \tag{1}$$

$$\frac{\partial^2 \theta_T}{\partial \xi^2}-A_2\theta_T-A_3\frac{\partial \theta_T}{\partial \tau}+A_4\frac{\partial^3 w}{\partial \xi^2\partial \tau}-A_5\frac{\partial^2 \theta_T}{\partial \tau^2}+A_6\frac{\partial^4 w}{\partial \xi^2\partial \tau^2}=0 \tag{2}$$

If the forcing term and the viscous effect of Eq. (1) are ignored, the governing equations become those given by Sun et al. [19]. Moreover, the beam is assumed very thin. Therefore, the temperature increment varies in terms of a $\sin(pz)$ function along the thickness direction, where $p=\pi/h$. based on the condition the second term of Eq. (2) is derived [19]. It is different to that given by Lifshitz and Roukes [16] where the upper and lower surfaces of beam cross-section at $z=\pm h/2$, are assumed adiabatic. In addition, the fifth and sixth terms of Eq. (2) are the effects of the phase lag for heat flux which are not considered in the Lifshitz and Roukes model.

The isothermal—isolated boundary conditions of a cantilever are:

At $\xi=0$

$$w(0,\tau)=0, \tag{3}$$

$$\frac{\partial w(0,\tau)}{\partial \xi}=0 \tag{4}$$

$$\theta_T(0,\tau)=0 \tag{5}$$

At $\xi=1$:

$$\frac{\partial^2 w(1,\tau)}{\partial \xi^2}=0, \tag{6}$$

$$\frac{\partial^3 w(1,\tau)}{\partial \xi^3}=0, \tag{7}$$

$$\theta_T'(1,\tau)=0. \tag{8}$$

Obviously, if the phase lag for heat flux $\tau_q=0$, the *C–V* model of heat conduction becomes the Fourier model. If the resonator is operating in fluid and close to the bounadry surface, the effect of the squeezing film damping should be significant. In the conventional, the squeezing film damping coefficient $c=c_s b^3/h_g^3$ where $c_s=\mu L^2/\sqrt{EI\rho A}$, [21].

3. Analytical solution

3.1. Solution method

The solutions of system composed of Eqs. (1)–(8) are assumed to be

$$\begin{aligned} w(\xi,\tau)&=w_c(\xi)\cos\,\omega\tau+w_s(\xi)\sin\,\omega\tau=\overline{w}\,\sin(\omega\tau-\varphi),\\ \theta_T(\xi,\tau)&=\theta_c(\xi)\cos\,\omega\tau+\theta_s(\xi)\sin\,\omega\tau=\overline{\theta}\,\sin(\omega\tau-\tilde{\varphi}), \end{aligned} \tag{9}$$

where

$$\overline{w}=\sqrt{w_c^2+w_s^2};\quad \varphi=\tan^{-1}\left(\frac{-w_c}{w_s}\right);$$
$$\overline{\theta}=\sqrt{\theta_c^2+\theta_s^2};\quad \tilde{\varphi}=\tan^{-1}\left(\frac{-\theta_c}{\theta_s}\right).$$

Substituting Eq. (9) into the Eqs. (1)–(8), one obtains

$$\frac{d^4 w_c}{d\xi^4}-\omega^2 A_1 w_c+c\omega w_s+\frac{d^2\theta_c}{d\xi^2}=0, \tag{10}$$

$$\frac{d^4 w_s}{d\xi^4}-\omega^2 A_1 w_s-c\omega w_c+\frac{d^2\theta_s}{d\xi^2}=f(\xi), \tag{11}$$

$$\frac{d^2\theta_c}{d\xi^2}+(A_5\omega^2-A_2)\theta_c-A_3\omega\theta_s+A_4\omega\frac{d^2 w_s}{d\xi^2}-A_6\omega^2\frac{d^2 w_c}{d\xi^2}=0, \tag{12}$$

$$\frac{d^2\theta_s}{d\xi^2}+(A_5\omega^2-A_2)\theta_s+A_3\omega\theta_c-A_4\omega\frac{d^2 w_c}{d\xi^2}-A_6\omega^2\frac{d^2 w_s}{d\xi^2}=0. \tag{13}$$

The corresponding boundary conditions are

At $\xi=0$:

$$w_c(0)=0, \tag{14}$$