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# Closed-form solution of the two-dimensional sliding frictional contact problem for an orthotropic medium

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ABSTRACT

The two-dimensional frictional contact problem of a rigid stamp sliding over an orthotropic medium is considered. The coordinate system is chosen such that it is aligned with the principal axes of orthotropy. It is parallel and perpendicular to the contact surface which is located along the $x_1$ axis. It is assumed that the condition of Coulomb friction prevails on the contact area. The two-dimensional half-plane problem is formulated using the Fourier integral transform method and the analytical formulation of the contact problem is reduced to a Cauchy-type singular integral equation of the second kind for the unknown contact pressure. The singular integral equation is solved analytically utilizing the Jacobi polynomials. With the application of the results to the crack initiation in an orthotropic medium in mind, the main emphasis in the study has been on the investigation of the singular nature of the stress state at either end of the flat stamp and on the determination of the contact stresses based on orthotropic material parameters. The present study provides the analytical solution of the contact stresses in terms of orthotropic material parameters, the coefficient of friction and the spatial coordinates. The strength of the singularities and the stress intensity factors at both ends of the stamp are also found in terms of the orthotropic material parameters and the coefficient of friction. This study reveals that orthotropic material parameters and the coefficient of friction have a great effect on the strength of stress singularities and distribution of the contact stresses. Adjusting these parameters will reduce these stresses that may have a bearing on the failure of the orthotropic medium. The results of this study will provide benchmark results for finite element analysts and stress engineers.



## 1. Introduction

Contact mechanics is an important part of the solid mechanics which investigates the stress distribution at the contacting surfaces of the mating parts in an assembly. In many of the engineering applications, loads are transferred through contacts between components, generally in the presence of friction (e.g. cylinder linings, brake discs, disc dove–tail connections and abradable seals in gas turbine components). In designing and especially estimating the service life of these components, it is critical to find the contact stresses since component failure usually occurs near the contact regions where the stress concentrations are high. These stresses may be responsible for component failure due to wear or fatigue.

Contact mechanics of a given structural component which is homogeneous and isotropic in its strength and thermomechanical properties is relatively well-understood and the techniques to deal with such problems are sufficiently well-established. On the other hand, in orthotropic materials such as composites, particularly in fiber-reinforced laminates, the situation is much more complicated mostly because of the large number of independent constants entering the analysis. Mechanical properties of orthotropic materials differ along each of the three principal axes of orthotropy and have two symmetry planes. These materials can be found in nature such as wood or intentionally developed such as fiber reinforced composites, thin films, crystals etc. They are more complex than isotropic materials since only two independent material constants are necessary to define the elastic properties of an isotropic material as opposed to nine in orthotropic materials.

The literature on the contact mechanics is quite large and dates back to Hertz [22]. The solution of the homogeneous isotropic contact mechanics problems are well established [33,46,34,47,28]. A through description of the problem can be found in the monographs [17,25,23,5] for homogeneous elastic materials. An extensive survey about contact mechanics problems can be found in the review paper by Barber and Ciavarella [4]. For the contact mechanics of graded materials, the readers are referred to

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[27,19]. The sliding frictional contact problem for a laterally graded medium is studied by Dag et al. [12]. Note that in all of the studies mention above, the medium is assumed to be isotropic whereas in this study the contacting medium is taken to be orthotropic.

However, contact mechanics of anisotropic materials are much less available. Some studies on the contact mechanics of transversely isotropic materials has been conducted by Green and Zerna [18], Leknitskii [30], Sveklo [49], Turner [52], Willis [51] and Dahan and Zarka [13]. Bakirtas [3] considered the frictionless contact problem of a rigid stamp on an elastic half plane with an orthotropic and non-homogeneous material. Hwu and Fan [24] solved the contact problems such as a frictionless flat punch indenting an orthotropic medium using the analogy with interface crack problem. In their paper, the analogy was made for the most general anisotropic linear elastic materials and the two-dimensional contact problems was solved directly by just using the corresponding solutions of crack problems.

Pan and Yuan [38] studied the three-dimensional Green's functions for the anisotropic bimaterials. Pan [39] formulated three-dimensional Green's functions in the anisotropic magneto–electro-elastic full space, half space and bimaterials based on the extended Stroh formalism. Later, Pan and Yang [40] derived the complete set of three-dimensional interfacial elastostatic Green's functions in the anisotropic bimaterials including displacements, stresses and their derivatives with respect to the source coordinates.

Borodich [9] solved some contact problems of the anisotropic elastodynamics by applying his method of integral characteristics of solutions to boundary–initial value problems. Ciavarella et al. [10] presented a method for solving 3D contact problem between generally anisotropic materials with any second order surface geometry. Their approach required the Fourier expansion of the Green's function for the frictionless normal contact problem which is found numerically using the efficient Barnett–Lothe method [6].

Shi et al. [45] presented the frictionless contact problem of a rigid ellipsoidal indenting an orthotropic half-space, with the surface of the half-space parallel to two of the axes of material symmetry. Swanson [50] calculated the stresses due to the contact loading in orthotropic materials by combining two previous solution techniques [51,41,48]. Lin and Ovaert [32] studied the two-dimensional isothermal rough surface contact problem of the general anisotropic materials with friction. Rand and Rovenskii [42] focused on the mathematical techniques and the solution methodologies required to establish the foundations of the anisotropic elasticity. Ning et al. [35] analyzed the problem of a rigid

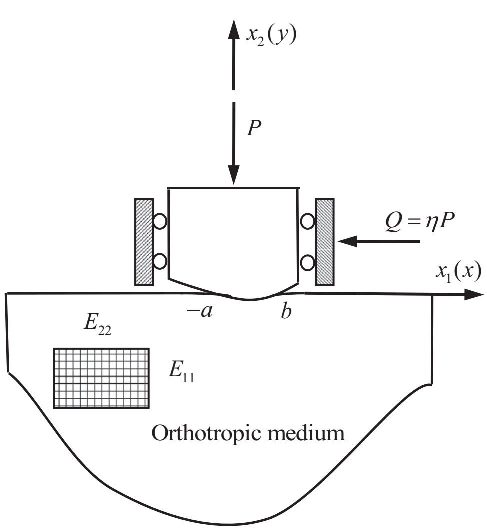


**Fig. 1.** Geometry of the sliding frictional contact problem.

sphere indenting a thin, transversely isotropic elastic layer resting on a rigid substrate. Li and Wang [31] studied the Hertzian contact problem of two generally anisotropic piezoelectric bodies using the Fourier transform method. The response of an arbitrarily multilayered piezoelectric orthotropic half-plane indented by a rigid frictionless parabolic punch was examined by Ramirez [43]. Galin [16] dealt with the contact problems of an anisotropic half plane. He and Ovaert [21] formulated a 3D rough surface contact problem for a semi-infinite anisotropic elastic half-plane in contact with a rough rigid sphere by applying the line integral of Barnett–Lothe tensors on oblique planes.

Batra and Jiang [7] used the Stroh formalism to analytically study the generalized plane strain deformations of a linear elastic anisotropic layer indented by a rigid cylindrical indenter. Jiang and Batra [26] employed the same methodology to study a generalized plane strain problem involving the indentation by a smooth rigid circular cylinder of a two-layer elastic composite with a through-the-width rectangular void between them.

Erbas et al. [14] investigated the pressure resulting from the frictionless contact of a punch with an elastic strip composed of an orthotropic material. The two dimensional frictionless sliding contact problem of orthotropic piezoelectric materials indented by a rigid punch sliding at a constant speed was investigated by Zhou and Lee [53]. Note that in the paper by Zhou and Lee [53], the two dimensional contact problem was solved under frictionless contact assumption therefore the only load acting to the orthotropic piezoelectric medium is the normally concentrated load $P$ and the tangential load, $Q$, acting on the contact area was taken to be zero (see Fig. 1). In this study we consider the frictional contact problem, therefore both the normally concentrated load $P$ and the tangential load, $Q$, are acting on the contact area. Bagault et al. [1,2] studied the effect of anisotropy orientation on the contact solution using semi-analytical methods relying on elementary analytical solution when an anisotropic half space is in contact with a rigid sphere. They developed a semi analytical method for the contact problem of anisotropic elastic materials with an anisotropic coating by using Green's functions.

The boundary element method was also used to solve the generalized plane problems of the anisotropic materials with possible friction contact zones (see [8] for 2D and [44] for 3D).

To the best of author's knowledge, two-dimensional frictional contact problem of a rigid stamp on an orthotropic medium has not been solved analytically in the open literature. The present study provides the analytical solution of the contact stresses in terms of orthotropic material parameters, the coefficient of friction and the spatial coordinates. The strength of the singularities and the stress intensity factors at both ends of the stamp are also found in terms of the orthotropic material parameters and the coefficient of friction.

The problem under consideration consists of a frictional sliding contact between an orthotropic homogeneous elastic half-plane and a rigid flat stamp subjected to the external loads $P$ and $Q$ (see Fig. 1). The mixed boundary value problem is solved analytically using the Fourier transform to convert the equations into a singular integral equation. An analytical closed-form solution is obtained for the contact stresses depending on the various material parameters and the coefficient of friction. The objective of the study is therefore to obtain a series of analytical benchmark solutions for examining the influence of material orthotropy and the coefficient of friction on the critical stresses that may have a bearing on the fatigue and fracture of the medium.

This paper is organized as follows. The problem description and formulation is provided in Section 2. The integral equation of the problem solution is detailed in Section 3. The analytical solution of the problem is given in Section 4. Since the values of the in-plane stress $\sigma_{11}$ on the surface are of particular interest in contact