International Journal of Mechanical Sciences 87 (2014) 241–257

Contents lists available at ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

Angles based integration for generalized non-linear plasticity model

Mohammad Rezaiee-Pajand [a,*], Mehrdad Sharifian [a], Mehrzad Sharifian [b]

[a] Ferdowsi University of Mashhad, Iran
[b] Quchan University of Advanced Technology, Quchan, Iran

ARTICLE INFO

Article history:
Received 18 December 2013
Received in revised form
9 June 2014
Accepted 11 June 2014
Available online 26 June 2014

Keywords:
Cyclic plasticity
Drucker-Prager's criterion
Nonlinear kinematic hardening
Reduced constitutive equations
Runge–Kutta methods
Exponential map

ABSTRACT

An effective integration method is proposed for a generalized nonlinear plasticity. The core of this study is to reduce the system of constitutive equations into a set of fewer scalar ones, which could be solved with a great many numerical integrations. The Optimal Implicit Strong Stability Runge–Kutta methods are suggested for this purpose due to their substantial features, such as precision, stability, and robustness. The qualities of the new approach are clearly discussed in a wide range of numerical tests comprising accuracy, efficiency, stability, and convergence rate assessments. Moreover, an initial boundary value problem is solved utilizing the proposed approach in practice. In addition to the implementation of the Optimal Implicit SSP Runge–Kutta methods, the Exponential Map integration is also advanced for the cyclic plasticity as a measure for the numerical tests, likewise, the Euler's integrations to conclude the study. The results demonstrate the superiority of the suggested technique.

1. Introduction

In a nonlinear elastoplastic finite element analysis, the material's behavior is contingent on the deformation history. This gives rise to incremental nonlinear equilibrium equations. Solving this set of nonlinear equations requires an iterative procedure, such as Newton–Raphson, in which strain histories are continually produced. Having these strain histories, the finite element method updates the stresses via integrating the constitutive equations. Basically, the constitutive laws provide relationships for the stresses in terms of the deformation history. These equations are typically nonlinear and complicated particularly when cyclic responses are due to consider. Therefore, nonlinear plasticity models are needed to precisely simulate the real behavior of the structure. The usage of nonlinear isotropic and kinematic hardening laws, such as those of the Chaboche, is strongly recommended in plasticity model to take account of ratcheting and transient stress–strain behavior of the material. These are two important phenomena most materials experience under cyclic loading [7]. Consequently, the predominant account in integrating the constitutive equations is the use of numerical integrations for the lack of analytical solutions to these problems. The accuracy and cost of the finite element analyses are closely related to the precision and robustness of the numerical integrations.

Many efforts have been carried out to develop more precise and efficient integrations over the years owing to the importance of the issue. Of the earliest attempts in this field, Wilkins [53] suggested the radial return-mapping integration. Rice and Tracy [42] used his technique to study the elastoplastic deformation of a crack tip proceeded by Krieg and Key [25] to take account of linear isotropic and kinematic hardenings. Later on, Krieg and Krieg [26] proposed an exact integration for the elastic perfectly plastic von-Mises plasticity. The main part of their work was to define an angle between the stress state and strain increment. Their method was extended by Schreyer et al. [44] to a plasticity with hardening. Later, Yoder and Whirley [55], Ortiz and Popov [33], and Runesson et al. [43] performed comparative investigations between different integration schemes showing the superiority of the return mapping algorithms. Meanwhile, Nagtegaal [32] first discerned the role of consistent tangent operators in achieving the quadratic convergence rate in implicit finite element codes which was followed by Simo and Taylor [45,46] and Dodds [12]. Two exact integrations were also proposed by Loret and Prevost [31] and Sloan and Booker [47] for Drucker–Prager [15] and Mohr–Coulomb plasticity models under certain circumstances, respectively. Afterwards, Genna and Pandolfi [16], Hopperstad and Remseth [20], and Wei et al. [54] presented integration schemes based on the radial return mapping and Prandtl–Reuss elastoplastic models. Hong and Liu [17–19] originally addressed the possibility of converting the constitutive equations into the concise form of $\dot{\mathbf{X}} = \mathbf{AX}$. This helped [6] introduce the Exponential Map integration. Meantime, Sloan et al. [48] integrated the constitutive

* Corresponding author. Tel./fax: +98 5118412912.
E-mail address: mrpajand@yahoo.com (M. Rezaiee-Pajand).

http://dx.doi.org/10.1016/j.ijmecsci.2014.06.009
0020-7403/

equations using an explicit Runge-Kutta technique with automatic error control. Kobayashi and Ohno [23] and Kobayashi et al. [24] used the closest projection method to offer implicit integrations for cyclic plasticity models. Subsequently, Liu [28–30] developed two exponential based integrations. Artioli et al. [3–5] and Rezaiee Pajand and Nasirai [34,35] improved the accuracy and convergence of the Exponential Map integrations. Later on, Clausen et al. [11] and Kan et al. [21] proposed two implicit integrations based on the return mapping strategies. Furthermore, expanding the procedure originally introduced by Krieg and Krieg [26], Wallin and Ristinmma [51,52,49], Szabó and Kossa [50] utilized Runge–Kutta methods to solve the constitutive laws. Afterwards, Rezaiee Pajand et al. [36–38] advanced the Exponential Map integrations to the von-Mises and Drucker–Prager plasticity models with linear isotropic and kinematic hardenings. They derived the consistent tangent moduli of their integrations in a recent study [40]. Moreover, they suggested two consistent exponential schemes for integrating nonassociative Drucker–Prager constitutive equations [41]. Meanwhile, Rezaiee et al. [39] introduced a new integrating algorithm defining an angle between the back stress and the stress state.

In the present study, it is intended to develop an integration scheme based on the works of Krieg and Krieg [26,39] for the nonlinear cyclic plasticity. The main difference, here, is to develop an integration method which takes account of generalized nonlinear kinematic and isotropic hardening laws. These are vitally important for simulating plastic strain accumulation, called ratcheting, and the stabilization of yield surface radius under cyclic loading. These become possible through Chaboche's nonlinear kinematic and isotropic hardening models which account for multiple back stress tensors along with a nonlinear function for the yield surface radius. Herein, a number of angles are defined between the components of the back stress tensor and the deviatoric stress. Thereby, the initial constitutive equations are reduced to much fewer differential equations. As a result, the new system could be solved through a wide variety of numerical integrations. As a new measure, the recently introduced Optimal Implicit Strong Stability Runge–Kutta methods are implemented to achieve great stability together with very precise updated stresses, even at exceptionally large load-step sizes. For the sake of comparison, the equations are also solved using the explicit Runge–Kutta methods alongside the exponential and classical Euler's integrations. The advantages of the suggested approach are demonstrated through numerical tests comprising diverse stress updating tests, stability investigation, and a typical initial boundary value problem.

2. Constitutive laws

A general plasticity model includes the following fundamental equations in the small strain realm:

$$F(\boldsymbol{\sigma},\ \mathbf{a})=0 \tag{1}$$

$$\sigma=\mathbb{D}^{e}:\varepsilon^{e} \tag{2}$$

$$\varepsilon=\varepsilon^{e}+\varepsilon^{p} \tag{3}$$

$$\dot{\varepsilon}^{p}=\dot{\gamma}\frac{\partial Q}{\partial\sigma},Q(\sigma,\ \mathbf{a}) \tag{4}$$

$$\dot{\boldsymbol{\kappa}}=-\dot{\gamma}\frac{\partial Q}{\partial\mathbf{a}},\mathbf{a}=-\rho\frac{\partial\psi^{p}}{\partial\boldsymbol{\kappa}} \tag{5}$$

$$\dot{\gamma}\geq 0,F\leq 0,\quad \dot{\gamma}F=0 \tag{6}$$

$$\dot{\sigma}=\mathbf{D}^{ep}\dot{\varepsilon} \tag{7}$$

Here, Eqs. (1) and (2) represent the yield function and the elastic law, respectively. Strain tensor is decomposed into elastic, ε^{e}, and plastic, ε^{p}, parts. The plastic flow rule is given by Eq. (4) where $\dot{\gamma}$ and Q stand for, respectively, plastic multiplier and plastic potential function in terms of the Cauchy stress, σ, and the hardening thermodynamic force, $\mathbf{a}$. Expressions in (5) provide a general description of hardening laws with $\boldsymbol{\kappa}$ representing the set of the hardening internal variables and $\rho\psi^{p}$ as the plastic part of *Helmholtz free energy* per unit mass. The plastic multiplier is obtained deploying the loading/unloading criterion (6):

$$\dot{\gamma}=\frac{\partial F/\partial\sigma:\mathbf{D}^{e}:\dot{\varepsilon}}{\partial F/\partial\sigma:\mathbf{D}^{e}:\partial Q/\partial\sigma+\partial F/\partial\mathbf{a}*D*\partial Q/\partial\mathbf{a}},\quad D=\rho\frac{\partial^{2}\psi^{p}}{\partial\kappa^{2}} \tag{8}$$

where $*$ symbolizes a proper product between $\partial F/\partial\mathbf{a}$, D and $\partial Q/\partial\mathbf{a}$. Utilizing the definition (7) together with the aforementioned equation, the elastoplastic tangent modulus reads:

$$\mathbf{D}^{ep}=\mathbf{D}^{e}-\frac{(\mathbf{D}^{e}:\partial F/\partial\sigma)\otimes(\mathbf{D}^{e}:\partial F/\partial\sigma)}{\partial F/\partial\sigma:\mathbf{D}^{e}:\partial Q/\partial\sigma+\partial F/\partial\mathbf{a}*D*\partial Q/\partial\mathbf{a}} \tag{9}$$

3. Plasticity model

In light of the general approach given, the constituents of the plasticity model considered in the study are written as:

$$F=\mathbf{s}'\mathbf{s}'-R^{2}=0;R=\sqrt{2}(\boldsymbol{\tau}_{y}-\boldsymbol{\beta}p')>0 \tag{10}$$

$$\dot{\mathbf{e}}^{p}=\dot{\boldsymbol{\gamma}}\mathbf{s}' \tag{11}$$

$$\dot{\varepsilon}_{v}^{p}=2\boldsymbol{\beta}\dot{\boldsymbol{\gamma}}(\boldsymbol{\tau}_{y}-\boldsymbol{\beta}p') \tag{12}$$

$$\dot{\boldsymbol{\tau}}_{y}=\overline{b}(\boldsymbol{\tau}_{y,0}+\boldsymbol{\tau}_{y,s}-\boldsymbol{\tau}_{y})\dot{\boldsymbol{\gamma}} \tag{13}$$

$$\dot{\alpha}=\sum_{i=1}^{m}\dot{\alpha}_{i},\quad \dot{\alpha}_{i}=H_{\mathrm{kin},i}\dot{\mathbf{e}}^{p}-H_{\mathrm{nl},i}\dot{\boldsymbol{\gamma}}\alpha_{i} \tag{14}$$

$$\dot{\overline{p}}=\sum_{i=1}^{m}\dot{\overline{p}}_{i},\dot{\overline{p}}_{i}=\frac{2}{3}H_{\mathrm{kin},i}\dot{\boldsymbol{\gamma}}\boldsymbol{\beta}(\boldsymbol{\tau}_{y}-\boldsymbol{\beta}p')-H_{\mathrm{nl},i}\dot{\boldsymbol{\gamma}}\overline{p}_{i} \tag{15}$$

The plasticity is a general model composed of Drucker–Prager yield criterion, Eq. (10), associative flow rule, Eqs. (11) and (12), together with the Chaboche's nonlinear isotropic and kinematic hardenings, Eqs. (12)–(15). Here, R represents the yield surface radius and $\boldsymbol{\tau}_{y}$ is the yield stress in pure shear. The variables $\mathbf{e}$, $\mathbf{s}'$, α and $\boldsymbol{\varepsilon}_{v}$, p', $\overline{p}$ designate the deviatoric and volumetric parts of the strain, ε, shifted stress, σ', and back stress, $\mathbf{a}$. These are concluded from the following decompositions:

$$\varepsilon=\mathbf{e}+\frac{\boldsymbol{\varepsilon}_{v}}{3}\mathbf{i};\quad \boldsymbol{\varepsilon}_{v}=\mathrm{tr}(\varepsilon) \tag{16}$$

$$\sigma'=\mathbf{s}'+p'\mathbf{i};\quad p'=\frac{\mathrm{tr}(\sigma')}{3} \tag{17}$$

$$\sigma=\sigma'+\mathbf{a} \tag{18}$$

$$\mathbf{a}=\alpha+\overline{p}\mathbf{i};\quad \overline{p}=\frac{\mathrm{tr}(\mathbf{a})}{3} \tag{19}$$

The subsequent expressions are formulated using Eqs. (2), (3) and (6) along with the aforementioned equations:

$$\dot{\mathbf{s}}'=2G\dot{\mathbf{e}}-2\overline{G}\dot{\boldsymbol{\gamma}}\mathbf{s}'+\sum_{i=1}^{m}H_{\mathrm{nl},i}\dot{\boldsymbol{\gamma}}\alpha_{i} \tag{20}$$

$$\dot{p}'=K\dot{\boldsymbol{\varepsilon}}_{v}-2\boldsymbol{\beta}\dot{\boldsymbol{\gamma}}\overline{\mathbf{K}}(\boldsymbol{\tau}_{y}-\boldsymbol{\beta}p')+\sum_{i=1}^{m}H_{\mathrm{nl},i}\dot{\boldsymbol{\gamma}}\overline{p}_{i} \tag{21}$$