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Two new hybrid methods in integrating constitutive equations

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ABSTRACT

It is crucial that more powerful integrations be developed as the plasticity models become more sophisticated. Here, two new robust integrating tactics are suggested based on the Exponential map and Euler's algorithms. The integrations are developed for the Drucker–Prager plasticity with nonlinear mixed hardening. Moreover, two different types of exponential strategies are advanced for the plasticity to be compared to the proposed techniques. Dealing with the apex of the yield surface is generally discussed for the integrations as well. Eventually, the proposed algorithms are thoroughly examined in a broad set of numerical tests comprising accuracy, efficiency, and convergence rate investigations. The results demonstrate the supremacy of the suggested schemes amid the six diverse techniques under discussion.

1. Introduction

Whenever a material undergoes a plastic deformation, the corresponding constitutive equations are path-dependent and as a result the stress tensor is contingent upon the strain history as well as function of the instantaneous amount of the strain. Assuming a general path-dependent plasticity model, integration of rate constitutive equations offers the updated stresses whose strain histories are produced in the iterative solution of the equilibrium equations during a generic nonlinear finite element analysis. As an essential necessity to the finite element simulation of path-dependent problems, an appropriate technique for integrating a constitutive law is of great significance due to being dramatically in touch with the final results of the numerical analysis.

The history of incremental theory of plasticity and the integration of elastoplastic constitutive equations, as an inseparable part of it, have their roots many years ago [1]. However, the current form of the most well-known integrating technique is initially attributed to Wilkins [2]. Having clearly explained the procedure of an entire elastoplastic analysis, he formulated the equation of State for two phases of elastic and plastic flow regions and introduced the radial return algorithm to update the stress tensor at elastoplastic state such that the yield strength of the material is not exceeded. His integration technique was pragmatically used by Rice and Tracy [3] to study the elastic perfectly plastic state of crack tip deformation by a finite element procedure. Later on, Krieg and Key [4] updated the algorithm for a more sophisticated plasticity to account for the isotropic and kinematic hardenings. Proposing an exact solution for perfect plasticity, Krieg and Krieg [5] are known as the frontiers of exact integrations. They also introduced the iso-error maps as a robust tool for examining the integrating tactics. Subsequently, Schreyer et al. [6] extended their scheme for a plasticity model with hardenings. At this point, it was time for a thorough investigation on the different integration schemes proposed to that time. Yoder and Whirley [7] performed a comparative examination which proved the general primacy of radial return integration, especially in the presence of hardening plasticity. Ortiz and Popov [8] extended the investigation on the ground of stability, likewise accuracy, to demonstrate the superior stability of the generalized midpoint over trapezoidal rules. An important fact of any of integration strategies is the consistent elastoplastic tangent modulus which is responsible for preserving the asymptotic rate of quadratic convergence in implicit finite element codes. The concept was first perceived by Nagtegaal [9] to which Simo and Taylor [10,11] and Dodds [12] are also known as pioneers. Meanwhile, Loret and Prevost [13] developed an exact solution for non-associative Drucker–Prager plasticity with linear isotropic hardening. After a brief review of new developments to that time, Runesson et al. [14] performed a comparison between midpoint and closest point integrations with respect to the treatment of non-smooth yield surfaces. Subsequently, Sloan and Booker [15] suggested that the constitutive equations of a non-smooth yield convex set such as Tresca or Mohr–Coulomb could be integrated exactly under certain circumstances. Afterwards, Genna and Pandolfi [16] illustrated a general two-step integration tactic

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for the associative rate plasticity of Drucker–Prager model with some advantages over the typical radial return algorithm. At this point, Hopperstad and Remseth [17] came by a modified return mapping algorithm that would employ the closest point algorithm along with a decomposition method to achieve a more efficient integration. The last major attempt to improve the integration schemes in twentieth century was carried out by Wei et al. [18] where they proposed a consistent algorithm based on Prandtl–Reuss elastoplastic models combining the advantages of exact time integration of constitutive equations and the quadratic asymptotic convergence of Newton–Raphson iterative procedures.

In the past century, as reviewed, the researches in this field were chiefly to develop the implicit group of integrations such as closest point, midpoint, trapezoidal and so on, which uses the unknown values of the variables at each load increment and leads to an iterative process in the heart of their algorithms. These studies were followed by Kobayashi and Ohno [19] and Kobayashi et al. [20] where they executed an integration strategy based on closest point method for a cyclic plasticity and presented an implicit method for time-and-temperature dependent constitutive equations. Afterwards, Clausen et al. [21] presented a new return method based on the constant gradient of linear isotropic yield criteria achieving simple formulae with closed form solution and no iteration for such criteria. Similarly, Kan et al. [22] proposed another implicit integration based on the radial return method and Backward Euler's integration. Coombs et al. [23] offered an alternative to the Drucker–Prager and Mohr–Coulomb models by a conical surface with modified Reuleaux deviatoric sections, which presents a better accuracy in stress updating with Backward Euler approach.

On the other side, there are explicit integrations to which there is no iterative part to update the stress state since they obtain the variables utilizing the known values of the preceding load step. In the recent century, this sort of integrations found a new life via inaugurating a new technique called Exponential Map integration. In essence, the strategy is based on converting the constitutive equations of a given plasticity into the compact form of $\dot{\mathbf{X}} = \mathbb{A}\mathbf{X}$, which was originally addressed by Hong and Liu [24–26]. Inspired by the notion, Auricchio and Beirao da Veiga [27] are regarded as the founders who practically presented an exponential-based algorithm for integrating the constitutive equations of von-Mises plasticity with linear mixed hardening. Subsequently, Artioli et al. [28] modified their algorithm to be consistent with the yield surface. Liu [29–31] investigated the internal symmetries of the elastic perfectly plastic Drucker–Prager model and developed two exponential-based integration schemes. Later on, accuracy and convergence rate of the method were improved by Artioli et al. [32,33] and Rezaiee Pajand and Nasirai [34,35]. Afterwards, Rezaiee Pajand et al. [36,37] demonstrated the application of exponential-based integrations for a class of von-Mises plasticity with nonlinear isotropic and kinematic hardenings. Their methods were all fast and consistent having a considerable accuracy and efficiency for the considered plasticity models.

Besides, there is another general category of explicit integrations where the constitutive equations are directly integrated by an explicit Runge–Kutta technique such as Forward Euler, 2-stage RK method of order 2. Of the most recent efforts in this field one could find Sloan et al. [38] and Solowski and Gallipoli [39] helpful. It is also possible to reduce the number of constitutive equations to fewer ODEs and then hire the explicit Runge–Kutta techniques for solving them. Many studies have been performed in this area from which Wallin and Ristinmma [40,41], Szabo [42], Kossa and Szabo [43] and Rezaiee Pajand and Sharifian [44] are the most notable ones in the recent century.

In this study, two robust hybrid schemes are proposed established upon three major integrations of Exponential map, Backward Euler, and Forward Euler. The new algorithms could easily handle sophisticated plasticity models where the aforementioned tactics unveil their weak spots. The Drucker–Prager yield criterion along with nonlinear isotropic and kinematic hardenings is chosen as the plasticity to generally develop the integrations and also challenge their capabilities. Coping with the apex of the yield surface is described for the two general approaches of explicit and implicit integrations, as well. Moreover, the exponential map algorithm, and also Eulers' are derived for the plasticity model, which in case of the exponential map is totally new concerning the plasticity. Eventually, in a comprehensive numerical examination, the suggested schemes are thoroughly investigated. First, the accuracy is scrutinized by means of strain histories and iso-error maps. Second, the efficiencies are inspected via putting the computational effort versus the accuracy. And third, using the stress relative errors for a succession of load-step increments, the convergence rates of the schemes are verified. In all numerical tests, the results are compared to those of the exponential and Eulers' to clearly demonstrate the great accuracy and efficiency of the suggested algorithms.

2. Constitutive models

To broadly develop the new hybrid techniques, a generic constitutive model is taken into account where the Drucker–Prager yield surface [45] represents a pressure dependent plasticity alongside nonlinear mixed hardening. Many other plasticity models could be particularized from this generic state. Small deformation realm is also assumed for the strains. The following is the yield function proceeded by Chaboche's nonlinear mixed hardening:

$$F = \frac{1}{2}\mathbf{s}'^{\mathrm{T}}\mathbf{s}' - (\tau_y - \beta p')^2 = 0, \quad \tau_y - \beta p' > 0 \tag{1}$$

$$\dot{\tau}_y = \overline{b}(\tau_{y,0} + \tau_{y,s} - \tau_y)\dot{\gamma} \tag{2}$$

$$\dot{\mathbf{a}} = \sum_{i=1}^{m} \dot{\mathbf{a}}_i, \quad \dot{\mathbf{a}}_i = H_{\mathrm{kin},i}\dot{\varepsilon}^{\mathrm{p}} - H_{\mathrm{nl},i}\dot{\gamma}\mathbf{a}_i \tag{3}$$

Eqs. (2) and (3) display, respectively, the nonlinear isotropic and kinematic hardenings where τ_y is the yield stress in pure shear, $\tau_{y,0}$ is the initial pure shear stress, and $\boldsymbol{\tau}_{y,s}$, $\overline{b}$, $H_{\mathrm{kin},i}$, and $H_{\mathrm{nl},i}$ are all material constants. For more in this, refer to Chaboche et al. [46], Chaboche [47,48] and Rezaiee-Pajand and Sinaie [49]. These hardening rules are the most applicable laws for regulating the yield surface evolution in nonlinear plasticity since they are quite capable of taking account of the transient stress–strain behavior of material and the ratcheting. Other parameters used in these relationships are all based on prevailing plasticity notation where a total stress is decomposed into its elements of shifted stress, $\boldsymbol{\sigma}'$, and back stress, $\mathbf{a}$:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \mathbf{a} \tag{4}$$

The parameters $\mathbf{s}'$ and p' are, respectively, the deviatoric and hydrostatic/volumetric parts of the shifted stress:

$$\boldsymbol{\sigma}' = \mathbf{s}' + p'\mathbf{i} \quad \text{with} \quad p' = \frac{\mathrm{tr}(\boldsymbol{\sigma}')}{3} \tag{5}$$

The presence of the Kinematic hardening means the existence of the back stress as it determines the yield surface center when its location is going to change. Analogous to the total stress, the associated back stress vector is divided into its deviatoric and hydrostatic parts, α and $\overline{p}$, respectively:

$$\mathbf{a} = \boldsymbol{\alpha} + \overline{p}\mathbf{i} \quad \text{with} \quad \overline{p} = \frac{\mathrm{tr}(\mathbf{a})}{3} \tag{6}$$

In a deviatoric space the nonlinear kinematic hardening finds the below shape:

$$\dot{\boldsymbol{\alpha}} = \sum_{i=1}^{m} \dot{\boldsymbol{\alpha}}_i, \quad \dot{\boldsymbol{\alpha}}_i = H_{\mathrm{kin},i}\dot{\mathbf{e}}^{\mathrm{p}} - H_{\mathrm{nl},i}\dot{\gamma}\boldsymbol{\alpha}_i \tag{7}$$