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Exact solutions of stresses, strains, and displacements of a perforated rectangular plate by a central circular hole subjected to linearly varying in-plane normal stresses on two opposite edges

Jae-Hoon Kang *

Chung-Ang University, 221 Heuksuk-Dong, Dongjak-Ku, Seoul 156-756, South Korea

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ABSTRACT

Exact solutions for stresses, strains, and displacements of a perforated rectangular plate by a central circular hole subjected to linearly varying in-plane normal stresses on two opposite edges are investigated by two-dimensional theory of elasticity using the Airy stress function. The hoop stresses, strains, and displacements occurring at the edge of the circular hole are computed and plotted. Comparisons are made for the stress concentration factors for several types of linearly varying in-plane loading.

1. Introduction

Numerous researchers have investigated the mechanical behaviors of perforated plates, with main concerns being classified into three categories; stress concentration [1–10], vibration [11–30], and buckling [31–60]. The various discrete methods have been used to study them. The finite element method [1–4,11–14,29,33,36,44,46–59] is the most widely used. Diverse methods other than the finite element method have been used like the complex variable method [5,6], three-dimensional stress analysis [7–10], the Ritz method [22–27,30], the boundary element method [15,16], the differential quadrature element method [17,18], semi-analytical solution method [19–22], experimental method [38], conjugate load/displacement method [43], and Galerkin averaging method [45]. Most of the shapes of perforated holes have three types of circular [1,4,7–9,11,31,32,36,38,44,46–48,50–60], elliptical [5,6,7,8,9,10,60], and rectangular cutout [5,6,22–27,30,36,38,43,44,46,47,49,54,57,58]. Exact solutions for perforated plates with a central circular hole loaded by linearly varying in-plane normal stresses have not been reported.

In the present study, exact solutions for stresses, strains, and displacements of a rectangular plate with a central circular hole subjected to linearly varying in-plane loading are investigated by two-dimensional theory of elasticity using the Airy stress function. The hoop stresses, strains, and displacements occurring at the edge of the circular hole are computed and plotted. Comparisons are made for the stress concentration factors for several types of linear varying in-plane normal stresses.

2. Method of analysis

Fig. 1 shows a rectangular plate of lateral dimensions $L \times h$ with a central circular hole of radius of a and subjected to linearly varying in-plane normal stresses on two opposite edges, and the rectangular (x,y) and polar (r,θ) coordinate systems. The plate is assumed to be large compared with the circular hole.

First of all, considering a rectangular plate with no hole subjected to linearly varying in-plane normal stresses, the stress components are

$$\sigma_{xx}^0 = \frac{\partial^2 \phi^0}{\partial y^2} = \frac{\sigma_0(1+\alpha)}{h} y + \frac{\sigma_0}{2}(\alpha - 1)$$
$$\sigma_{yy}^0 = \frac{\partial^2 \phi^0}{\partial x^2} = 0$$
$$\sigma_{xy}^0 = -\frac{\partial^2 \phi^0}{\partial x \partial y} = 0 \qquad (1)$$

where ϕ^0 is a fundamental Airy stress function, σ_0 is the intensity of compressive stress at $y = -h/2$, and α is a numerical loading

* Tel.: +82 2 820 5342; fax: 82 2 812 4150.
E-mail address: jhkang@cau.ac.kr

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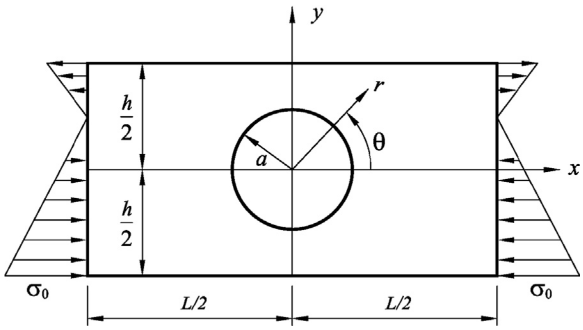

Fig. 1. A rectangular plate perforated by a central circular hole loaded by linearly varying in-plane loading on two opposite edges.

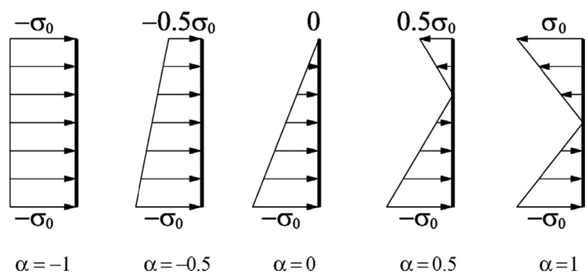

Fig. 2. Examples of in-plane loading σ_{xx} along the edge $x=-L/2$.

factor. By changing α, we can obtain various particular cases. For example, by taking $\alpha=-1$ we have the case of uniformly distributed compressive force. When $\alpha=0$, the compressive force varies linearly from $-\sigma_0$ at $y=-h/2$ to zero at $y=h/2$. For $\alpha=1$ we obtain the case of pure in-plane bending moment. With other α in the range $-1<\alpha<1$, we have a combination of bending and compression. Examples of these cases are shown in Fig. 2. For $\alpha<-1$ or $\alpha>1$ the problems arising are identical with ones having $-1<\alpha<1$.

The Airy stress function ϕ satisfies the governing equation $\nabla^4\phi=\nabla^2(\nabla^2\phi)=0$ with no body forces in 2-D plane problems in elasticity, where the Laplacian differential operator ∇^2 is expressed as

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{2}$$

and ∇^4 is the bi-harmonic differential operator defined by

$$\nabla^4=\nabla^2(\nabla^2)=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4} \tag{3}$$

in the rectangular coordinates. From Eq. (1), the fundamental Airy function ϕ^0 can be assumed as

$$\phi^0=\frac{\sigma_0(1+\alpha)}{6h}y^3+\frac{\sigma_0}{4}(\alpha-1)y^2+Ay+Bx+C \tag{4}$$

where A, B, and C are arbitrary integration constants. A linear function of x or y and a constant in the Airy stress function are trivial terms which do not give rise to any stresses and strains. Dropping the trivial terms in Eq. (4), the fundamental Airy stress function ϕ^0 becomes

$$\phi^0=\frac{\sigma_0(1+\alpha)}{6h}y^3+\frac{\sigma_0}{4}(\alpha-1)y^2 \tag{5}$$

Using the relation of

$$y=r\sin\theta \tag{6}$$

Table 1
Stresses of potential candidates of bi-harmonic functions ϕ.

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\ln r$	$1/r^2$	0	$-1/r^2$
$r^2\ln r$	$2\ln r+1$	0	$2\ln r+3$
$r^3\sin\theta$	$2r\sin\theta$	$-2r\cos\theta$	$6r\sin\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$-\cos\theta/r$	$\sin\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\cos\theta/r^3$	$2\sin\theta/r^3$
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^4\cos 2\theta$	0	$6r^2\sin 2\theta$	$12r^2\cos 2\theta$
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0
$r^3\sin 3\theta$	$-6r\sin 3\theta$	$-6r\cos 3\theta$	$6r\sin 3\theta$
$r^5\sin 3\theta$	$-4r^3\sin 3\theta$	$-12r^3\cos 3\theta$	$20r^3\sin 3\theta$
$\sin 3\theta/r^3$	$-12\sin 3\theta/r^5$	$12\cos 3\theta/r^5$	$12\sin 3\theta/r^5$
$\sin 3\theta/r$	$-10\sin 3\theta/r^3$	$6\cos 3\theta/r^3$	$2\sin 3\theta/r^3$

and the multiple angle formulas

$$\sin^2\theta=\frac{1-\cos 2\theta}{2} \tag{7}$$

$$\sin^3\theta=\frac{3\sin\theta-\sin 3\theta}{4} \tag{8}$$

Eq. (5) can be transformed into the bi-harmonic functions as

$$\phi^0=\frac{\sigma_0}{24}\left[\frac{(1+\alpha)}{h}(3r^3\sin\theta-r^3\sin 3\theta)-3(\alpha-1)(r^2\cos 2\theta-r^2)\right] \tag{9}$$

which satisfies the governing equation $\nabla^4\phi^0=\nabla^2(\nabla^2\phi^0)=0$, where the Laplacian differential operator ∇^2 is

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{10}$$

and the bi-harmonic differential operator ∇^4 is expressed as

$$\nabla^4=\nabla^2(\nabla^2)=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{11}$$

in the polar coordinates. From the relation between stresses and the Airy stress function in the polar coordinates, the stress components in the rectangular plate *with no hole* subjected to linearly varying in-plane normal stresses can be calculated as below:

$$\sigma_{rr}^0=\frac{1}{r}\frac{\partial\phi^0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi^0}{\partial\theta^2}=\frac{\sigma_0}{4}\left\{\frac{1+\alpha}{h}(\sin\theta+\sin 3\theta)\ r+(\alpha-1)(\cos 2\theta+1)\right\} \tag{12}$$

$$\sigma_{r\theta}^0=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi^0}{\partial\theta}\right)=\frac{\sigma_0}{4}\left\{\frac{1+\alpha}{h}(\cos 3\theta-\cos\theta)\ r+(1-\alpha)\sin 2\theta\right\} \tag{13}$$

Let us return to the original problem of a *perforated* rectangular plate by a central circular hole. The total Airy function ϕ becomes

$$\phi=\phi^0+\phi^* \tag{14}$$

where ϕ^* is an Airy stress function to cancel unwanted traction due to ϕ^0 on $r=a$. The normal and shear stresses on $r=a$ must be free as below

$$\sigma_{rr}|_{r=a}=[\sigma_{rr}^0+\sigma_{rr}^*]_{r=a}=0 \tag{15}$$

$$\sigma_{r\theta}|_{r=a}=[\sigma_{r\theta}^0+\sigma_{r\theta}^*]_{r=a}=0 \tag{16}$$

Therefore, σ_{rr}^* and $\sigma_{r\theta}^*$ on $r=a$ must have terms of $\sin\theta$, $\sin 3\theta$, $\cos 2\theta$ or a constant and have $\cos\theta$, $\cos 3\theta$, or $\sin 2\theta$, respectively, in order to eliminate the stresses on $r=a$ due to ϕ^0. Tables 1 and 2