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Numeric-analytic solutions of the smooth and discontinuous oscillator

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ABSTRACT

Earlier works on the smooth and discontinuous (SD) oscillator concentrated mainly on the time domain analysis using analytical, semi-analytical and numerical integration methods. In this paper, the frequency domain analysis of the SD oscillator subjected to harmonic excitation which is as important and giving further insight into the dynamics is carried out. Multi-Harmonic Balance Method (MHBM) in combination with arc length continuation is used to obtain the periodic solutions and their branches in the frequency domain for different values of the smoothing parameter α and exciting frequency ω. Stability of the periodic motions and bifurcation behavior are analyzed using the Floquet theory. For the discontinuous case, the oscillator is treated as a Filippov system and an event driven numerical integration method is used to obtain the response. For $\alpha > 1$, the dynamics of the SD oscillator is similar to that of the hardening Duffing oscillator, for $\alpha = 1$, it is like that of the Ueda oscillator and for $0 < \alpha < 1$ it is like that of the Duffing oscillator with double well potential. The SD oscillator exhibits period 1 solutions, higher order periodic solutions, chaotic solutions through symmetry breaking bifurcations, period doubling and boundary crises in different parameter ranges. Chaos is observed over a larger frequency range interspersed by narrow windows of higher order periodic solutions.

1. Introduction

The dynamics of harmonically excited Duffing oscillator [1] and Van der Pol oscillator [2] have been studied extensively in the literature. The nonlinear terms in these oscillators are continuous in nature. These oscillators exhibit periodic solutions of different orders, subharmonic and superharmonic solutions and chaotic motions depending on the parameter values and initial conditions. Other oscillators with continuous nonlinearities have also been studied in the literature [3,4]. Approximate methods such as perturbation method, averaging method, method of multiple scales and harmonic balance method have been used to obtain the periodic motions of such systems.

Nonlinearities in physical systems such as the rotor stator interactions in gas turbines, dry friction damping as in the case of gas turbine blades with underplatform dampers and shrouded blade system, complex frictional contact and backlash as in the case of gear tooth, intermittent separation of tool and workpiece in metal cutting, switching in electric circuits and hybrid dynamics in control devices are discontinuous in nature. The differential equations representing these systems are discontinuous in nature. These systems are classified into hybrid systems, Filippov systems, piecewise smooth continuous systems and systems with higher order nonlinearities [5]. These nonlinear systems when subjected to harmonic excitation exhibit periodic motions of different orders, chaotic motions and coexistence of a number of different solutions. Such systems also exhibit discontinuity induced bifurcations such as sliding, grazing and border collision bifurcations [6].

Analytical, approximate analytic and numeric-analytic methods have been used to study the dynamics of nonlinear oscillators with discontinuous nonlinearities. One method to solve the discontinuous dynamical systems is by replacing the discontinuous nonlinearity by a smoothing function so that the system can be solved as a continuous nonlinear system. Narayanan and Jayaraman [7] studied the vibration in a nonlinear oscillator with Coulomb damping. The sgn (signum) function is approximated by an arc tangent function to obtain the periodic and chaotic motions of the system. Kim et al. [8] studied the effect of smoothing functions on the frequency response of an oscillator with clearance nonlinearity. The disadvantage of using smoothing functions is that the differential equations become stiff and are computationally expensive and sometimes the original dynamics of the system will be lost during the smoothing procedure. Wiercigroch [9] represented some discontinuous systems as approximate continuous systems in a number of continuous subspaces. The global solution is thus obtained by gluing together the local solutions in each subspace. The Filippov method is used to solve dynamic systems with discontinuous nonlinearities [10]. Event driven methods, switch model and time stepping methods [11] are used for the solution of Filippov systems. Apart from numerical integration based methods

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other semi-analytical methods based on harmonic balance and its variants are also used for the solution of dynamic systems with discontinuous nonlinearities [12].

Cao et al. [13] proposed an archetypal model of an oscillator for smooth and discontinuous dynamics. They named it as SD oscillator. The nonlinearity associated with the oscillator is with reference to its geometric configuration. The nonlinearity can be classified as smooth or discontinuous depending on the value of a non-dimensional parameter termed as smoothness parameter (α). Physically it represents a snap through truss system.

Cao et al. [13] investigated the dynamics of the SD oscillator by considering both continuous and discontinuous nature of the oscillator. The equilibrium points of the undamped unforced oscillator are obtained and classified with respect to the nature of the solution around the equilibrium for different values of the smoothness parameter α ($\alpha = 0, 0 < \alpha < 1, \alpha = 1$ and $\alpha > 1$). The system has similarities to the double well potential of the Duffing system for $0 < \alpha < 1$. The discontinuous oscillator has a saddle like singularity associated with it. Bifurcation diagrams are generated with α as the parameter for the viscously damped harmonically excited system. The system has coexisting periodic solutions of various orders and chaotic solutions. The existence of chaotic transients leading to periodic solutions for smaller values of α is explained. Further investigation on the discontinuous oscillator is presented in [14]. Even though the physical configuration in the discontinuous case is not possible, a conceptual model capable of producing rich dynamic behavior is established. The KAM (Kolmogorov–Arnold–Moser) structure of the undamped forced system is generated. It shows a chaotic sea with quasiperiodic islands. Higher order periodic solutions are also observed. With the addition of damping, these quasiperiodic solutions are changed to the corresponding periodic solutions. Bifurcation diagrams generated show higher order periodic solutions, period doubling route to chaos and chaotic transients. A semi-analytical method developed in [15] is used for the numerical solution to avoid numerical difficulties associated with the discontinuous system. This work considers a piecewise linear approximation of the SD oscillator. Both smooth and discontinuous systems are investigated with this approximation. Close agreement in behavior between the original system and the piecewise linear system is observed.

The Hopf bifurcation of the SD oscillator near the equilibrium points is investigated in [16] with an addition of nonlinear damping and external excitation for both $\alpha > 0$ and $\alpha = 0$. Poincare–Birchoff normal forms are obtained based on which the Hopf bifurcations are investigated. For the trivial equilibrium point, Hopf bifurcation exists only for $\alpha > 1$ and non-Hopf periodic solutions exist for $0 \le \alpha < 1$. For the nontrivial equilibria, Hopf bifurcation exists for $0 < \alpha < 1$ and non-Hopf bifurcation periodic behavior for $\alpha = 0$. The saddle like singularity found near the trivial equilibrium point changes to a periodic solution with the variation of the parameters. The co-dimension of two bifurcation of the oscillator is investigated in [17]. For $\alpha = 1$, the eigenvalues associated with the trivial equilibrium point are zero and the equilibrium point is non-hyperbolic. The bifurcation diagram in the parameter space $\alpha - \zeta$ plane and the corresponding co-dimension-two behavior near the trivial equilibrium for $\alpha = 1$ is obtained.

Periodic solutions of the viscously damped harmonically excited SD oscillator are obtained using an extended averaging method [18]. The stability of the solutions is investigated using the Lyapunov method. The solution diagrams are generated for different values of the forcing amplitude. Both stable and unstable solutions are obtained. Stable solutions are validated numerically.

Recently Cao et al. [19] analytically investigated the periodic and homoclinic solutions of the SD oscillator using irrational elliptic and hyperbolic functions. The chaotic threshold is found out using Melnikov functions with respect to the hyperbolic functions. Other relevant works on the SD oscillator include the analysis of the oscillator under constant excitation which leads to the loss of symmetry [20]. The chaotic attractor in this case for both smooth and discontinuous cases is significantly different from that without constant excitation. The threshold of the constant excitation for chaos for the SD oscillator is discussed in [21].

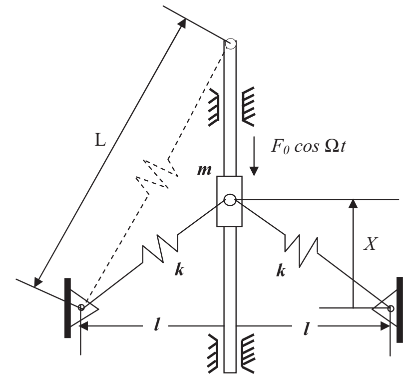

Fig. 1. SD oscillator (redrawn from [13]).

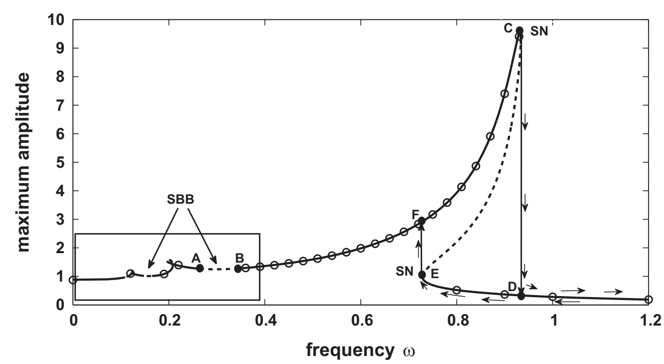

Fig. 2. Frequency response for $\alpha = 1.1$, $f_0 = 0.25$, $\zeta = 0.0141$ (____ stable P1, - - - - unstable P1 and ○ NI).

The SD oscillator representing the snap through truss system has been recently used as a nonlinear vibration absorber in [22] and also as an energy harvesting mechanism [23].

The above studies are mainly based on the time integration method and also on the dependence of the parameter α on the response. Frequency response analysis of the oscillator can give further and important information of the dynamic behavior of the nonlinear oscillator subjected to harmonic excitation. The frequency response analysis of the SD oscillator has not been treated adequately in the literature. This paper presents the dynamics of the SD oscillator in the frequency domain by systematically investigating the periodic solutions and the associated bifurcations providing more insight into the rich dynamics of the oscillator. The periodic solutions of the SD oscillator subjected to harmonic excitation are obtained by the multi-harmonic balance method (MHBM) [24–26]. In the MHBM the periodic solution is expressed as a truncated Fourier series and substituted in the equations of motion yielding a residual which is minimized. Subsequent application of the Galerkin and Newton–Raphson method leads to the determination of the Fourier coefficients and the periodic solutions of the SD oscillator. The frequency domain method is efficiently combined with an arc length continuation technique to follow a particular periodic solution branch for parameter variations. Stability analysis based on the Floquet theory [27] is used to determine the stability of the periodic solutions and the associated bifurcations of the SD oscillator. Both stable and