International Journal of Mechanical Sciences 73 (2013) 40–52

Contents lists available at SciVerse ScienceDirect

International Journal of Mechanical Sciences

journal homepage: www.elsevier.com/locate/ijmecsci

# A simple refined theory for bending, buckling, and vibration of thick plates resting on elastic foundation

Huu-Tai Thai, Minwo Park, Dong-Ho Choi *

*Department of Civil and Environmental Engineering, Hanyang University, 17 Haengdang-dong, Seongdong-gu, Seoul 133-791, Republic of Korea*

ARTICLE INFO

*Article history:*
Received 22 November 2011
Received in revised form
12 February 2013
Accepted 29 March 2013
Available online 24 April 2013

*Keywords:*
Analytical solution
Bending
Buckling
Vibration
Plate theory
Elastic foundation

ABSTRACT

A simple refined shear deformation theory is proposed for bending, buckling, and vibration of thick plates resting on elastic foundation. The theory accounts for parabolic distribution of transverse shear stress, and satisfies the free transverse shear stress conditions on the top and bottom surfaces of the plate without using shear correction factor. The number of unknowns of present theory is two as against three in the case of other shear deformation theories. The elastic foundation is modeled as two-parameter Pasternak foundation. Equations of motion are derived from Hamilton's principle. Analytical solutions are obtained for rectangular plates with two opposite edges simply supported and the other two edges having arbitrary boundary conditions. Comparison studies are presented to verify the validity of present solutions. It can be concluded that the proposed theory is accurate and efficient in predicting the bending, buckling, and vibration responses of thick plates resting on elastic foundation.



## 1. Introduction

The bending, buckling, and vibration problems of plates on elastic foundation have attracted the attention of many researchers working on structural foundation analysis and design. To describe the interaction between plate and foundation, various kinds of foundation models have been proposed. The simplest one is Winkler [1] which regards the foundation as a series of separated spring without coupling effects between each other. This model was improved by Pasternak [2] by adding a shear spring to simulate the interactions between the separated springs in the Winkler model. The Pasternak or two-parameter model is widely used to describe the mechanical behavior of structure–foundation interactions.

Extensive studies about plates on elastic foundation can be found in the literature. All of these studies were based on classical plate theory [3–8], first-order shear deformation theory [9–30], higher-order shear deformation theory [31–34], and three-dimensional elasticity theory [35–37]. It should be noted that the classical plate theory (CPT) is applicable to thin plate. For moderately thick plates, it underestimates deflection and overestimates buckling load and natural frequency due to ignoring the transverse shear deformation effects. The first-order shear deformation theory (FSDT) accounts for the transverse shear deformation effects, but requires a shear correction factor to satisfy the free transverse shear stress conditions on the top and bottom surfaces of the plate. Although the FSDT provides a sufficiently accurate description of response for thin to moderately thick plates, it is not convenient to use due to difficulty in determination of correct value of the shear correction factor. Hence, higher-order shear deformation theories (HSDTs) were proposed to avoid the use of shear correction factor and obtain better prediction of response of thick plate. Since the HSDTs are based on assumption of quadratic, cubic or higher-order variations of in-plane displacements through the thickness, their equations of motion are much more complicated than those of FSDT. Hence, there is a scope to develop an accurate theory which is simple to use.

The paper aims to propose a refined shear deformation theory for bending, buckling, and vibration of plates on elastic foundation which is simple to use. This theory is based on assumption that the in-plane and transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The most interesting feature of this theory is that it contains two unknowns as against three in the case of other shear deformation theories. In addition, it does not require shear correction factor, and has strong similarities with the CPT in some aspects such as equations of motion, boundary conditions, and stress resultant expressions. The elastic foundation is modeled as two-parameter Pasternak foundation. Equations of motion are derived using Hamilton's principle.

* Corresponding author. Tel.: +82 2 2220 0328; fax: +82 2 2220 4322.
*E-mail addresses:* taispkt@yahoo.com (H.-T. Thai), samga@hanyang.ac.kr (D.-H. Choi).

0020-7403/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijmecsci.2013.03.017

Analytical solutions are obtained for rectangular plates with two opposite edges simply supported and the other two edges having arbitrary boundary conditions. Numerical examples are presented to verify the accuracy of the proposed theory in predicting the deflection, buckling load, and natural frequency of plates resting on elastic foundation. Finally, some new results are presented to serve as references for comparison with further plate models.

## 2. Theoretical formulations

### 2.1. Basic assumptions

Consider a rectangular plate with length $a$, width $b$, and thickness $h$ resting on Pasternak foundation as shown in Fig. 1. The assumptions of the present theory are as follows:

i. The displacements are small in comparison with the plate thickness and, therefore, strains involved are infinitesimal.
ii. The transverse normal stress $\sigma_z$ is negligible in comparison with in-plane stresses $\sigma_x$ and $\sigma_y$.
iii. The transverse displacement $u_3$ includes two components of bending $w_b$ and shear $w_s$. These components are functions of coordinates $x$, $y$, and time $t$ only.

$$u_3(x,y,z,t) = w_b(x,y,t) + w_s(x,y,t) \tag{1}$$

iv. The in-plane displacements $u_1$ and $u_2$ consist of bending and shear components.

$$u_1 = u_b + u_s \text{ and } u_2 = v_b + v_s \tag{2}$$

– The bending components $u_b$ and $v_b$ are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for $u_b$ and $v_b$ can be given as

$$u_b = -z\frac{\partial w_b}{\partial x} \text{ and } v_b = -z\frac{\partial w_b}{\partial y} \tag{3a}$$

– The shear components $u_s$ and $v_s$ give rise, in conjunction with $w_s$, to the parabolic variations of shear strains $\gamma_{xz}$, $\gamma_{yz}$ and hence to shear stresses $\sigma_{xz}$, $\sigma_{yz}$ through the thickness $h$ of the plate in such a way that shear stresses $\sigma_{xz}$, $\sigma_{yz}$ are zero at the top and bottom surfaces of the plate. Consequently, the expression for $u_s$ and $v_s$ can be given as

$$u_s = \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial x} \text{ and } v_s = \left[\frac{1}{4}z - \frac{5}{3}z\left(\frac{z}{h}\right)^2\right]\frac{\partial w_s}{\partial y} \tag{3b}$$

### 2.2. Kinematics

Based on the assumptions made in the preceding section, the displacement field of the proposed theory can be obtained using Eqs. (1)–(3) as

$$\begin{aligned} u_1(x,y,z,t) &= -z\partial w_b/\partial x - f\partial w_s/\partial x \\ u_2(x,y,z,t) &= -z\partial w_b/\partial y - f\partial w_s/\partial y \\ u_3(x,y,z,t) &= w_b(x,y,t) + w_s(x,y,t) \end{aligned} \tag{4}$$

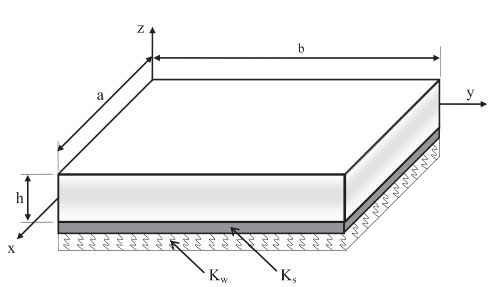


**Fig. 1.** Geometry and coordinate of rectangular plate resting on elastic foundation.

where $f = -z/4 + 5z^3/3h^2$. The linear strain can be obtained from kinematic relations as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = z\begin{Bmatrix} \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \end{Bmatrix} + f\begin{Bmatrix} \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g\begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} \tag{5}$$

where

$$\begin{Bmatrix} \kappa_x^b \\ \kappa_y^b \\ \kappa_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_b}{\partial x^2} \\ -\frac{\partial^2 w_b}{\partial y^2} \\ -2\frac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \kappa_x^s \\ \kappa_y^s \\ \kappa_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w_s}{\partial x^2} \\ -\frac{\partial^2 w_s}{\partial y^2} \\ -2\frac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{xz}^s \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_s}{\partial y} \\ \frac{\partial w_s}{\partial x} \end{Bmatrix},$$

$$g = 1 - \frac{df}{dz} = \frac{5}{4} - 5\left(\frac{z}{h}\right)^2 \tag{6}$$

### 2.3. Constitutive equations

For an isotropic plate, the constitutive relations can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{7}$$

where $E$ and $\nu$ are the Young's modulus and Poisson's ratio of plate, respectively.

### 2.4. Equations of motion

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in analytical form as [38]

$$0 = \int_0^T (\delta U_P + \delta U_F + \delta V - \delta K)dt \tag{8}$$

where $\delta U_P$ and $\delta U_F$ are the variation of strain energy of the plate and foundations, respectively; $\delta V$ is the variation of potential energy of the applied loads; and $\delta K$ is the variation of kinetic energy of the mass system.

The variation of strain energy of the plate can be stated as

$$\begin{aligned} \delta U_P &= \int_V (\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_{xy}\delta\gamma_{xy} + \sigma_{yz}\delta\gamma_{yz} + \sigma_{xz}\delta\gamma_{xz})dAdz \\ &= \int_A \Bigg\{ -M_x^b\frac{\partial^2 \delta w_b}{\partial x^2} - M_x^s\frac{\partial^2 \delta w_s}{\partial x^2} - M_y^b\frac{\partial^2 \delta w_b}{\partial y^2} - M_y^s\frac{\partial^2 \delta w_s}{\partial y^2} \\ &\quad -2M_{xy}^b\frac{\partial^2 \delta w_b}{\partial x \partial y} - 2M_{xy}^s\frac{\partial^2 \delta w_s}{\partial x \partial y} + Q_{yz}\frac{\partial \delta w_s}{\partial y} + Q_{xz}\frac{\partial \delta w_s}{\partial x}\Bigg\}dA \end{aligned} \tag{9}$$

where $M$ and $Q$ are the bending moment and transverse shear force defined as

$$(M_i^b, M_i^s) = \int_{-h/2}^{h/2}(z,f)\sigma_i dz, (i = x, y, xy) \text{ and } Q_i = \int_{-h/2}^{h/2} g\sigma_i dz (i = xz, yz), \tag{10}$$

The variation of strain energy of the foundation can be expressed as

$$\delta U_F = \int_A \left[K_w u_3 \delta u_3 + K_s\left(\frac{\partial u_3}{\partial x}\frac{\partial \delta u_3}{\partial x} + \frac{\partial u_3}{\partial y}\frac{\partial \delta u_3}{\partial y}\right)\right]dA \tag{11}$$