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Can Coulomb criterion be generalized in case of ductile materials? An application to Bridgman experiments

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The failure behavior of isotropic non-linear elastic materials is macroscopically studied in terms of elastic strain energy density generalizing the Coulomb criterion. This generalization is based on a rigorous mathematical substrate developed on the principle of conservation of the total elastic strain energy. In the general case of loading the behavior of a material is described with regard to the secant elastic moduli depending on both first strain and second deviatoric strain invariants. This dependence enlightens, in physical terms, the different reaction of materials in normal and shear stresses. Besides, these two moduli establish two constitutive equations for the complete description of any material, instead of the usual one. A theoretical application is given and the failure surfaces which are obtained in stress space are being commented. Predictions obtained in tension of steel under pressure from Bridgman's experiments and some of his observations for the failure behavior of steels are explained on the existence of a universal criterion with the present approach.

1. Introduction

The development of failure criteria outside Fracture Mechanics in either macro- or micro-scale seems, nowadays, out of fashion. Cracks and other geometrical singularities are considered as obligatory ingredients for a modern failure criterion. Does it imply that cracks affect, somehow, the *character* (the mechanical behavior) of materials so strongly as to necessitate the development of a class of criteria not applicable in absence of cracks? In addition, microscopic approaches are considered more fruitful, due to the necessity of developing and assessing new materials. Thus, a high barrier was raised between Fracture Mechanics and classical Strength of Materials. Sometimes the situation becomes absurd when "accidentally" a pre-cracked specimen (belonging nominally to Fracture Mechanics) fails at the loading machine grips (belonging nominally to Strength of Materials) instead of the "correct" point of crack-tip. Which family of criteria is proper for the description of this failure? Who decided that this pre-cracked specimen was "heretical"?

Failure is the transition of a virgin or intact elementary volume of the elastic material into a different irreversible state called "failure". Any geometrical singularity *per se* is already a local failure and acts as the cause/trigger of a next failure. Nuclei of local failure are always created and failure develops not only in the close area of crack-tips but also at machine grips (scratches, punches, etc.) or even at invisible geometrical singularities inside of material. Usually one of them drives to global failure. **This must happen under the application of the same rules**. In other words, a reasonable criterion must work equally well in Fracture Mechanics and classical Strength of Materials. Otherwise, a non-scientific and casuistic approach is adopted.

Apparently, local failure starting everywhere in the specimen may stop or integrate to global failure depending on the material and the available (i.e. elastic) strain energy. Strain energy is a good (if not the sole) candidate for controlling failure since it is an objective and smooth measure of all components of stress/strain fields. The necessity of the introduction of strain energy into failure criteria was clearly understood so early as at the era of Maxwell [1] and others [2,3]. Furthermore, a reasonable failure criterion must contain at least one material property independent of specimen geometry and other mechanical parameters indirectly connected to geometry and loading system. If so, peculiar beliefs like different critical values of stress intensity factors in plane stress and plane strain states reduce to nonsense.

The need of availability of strain energy excludes *ab initio* the involvement of plastic strains/work in failure. **Plastic strains/work is an irreversible *result* not a *cause***. Thus, special attention must be given to plastic work because it is strongly connected to specimen geometry. Any criterion using as its critical quantity a function depending on plastic work is casuistic. In fact, during

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loading a part of the total strain energy is converted irreversibly to plastic work, no more available for future use including the cost of failure. It is true that elastic and plastic strains interact mutually depending on geometry and loading path and, consequently, failure may be observed at different combinations of elastic strains. However, the final result is that *elastic* strain energy density has some very interesting properties at the moment of failure, as it will be evident in the sequel.

Some basic assumptions may be worthy at this point:

1. Practical applications require answers in terms of macroscopic level, i.e. Continuum Mechanics. Microscopic considerations are intentionally excluded here, for an additional reason that they, usually, refer to a particular material type [4].
2. Constitutive equations need to give an insight to the nature of a material and to be as much as possible adequate and simple to describe stresses/strains relations.
3. Cracks and other geometrical singularities must be solely understood as modulators of the stress/strain fields and nothing more.
4. Whatever happens in a material under loading must be supported by *available* (i.e. elastic) strain energy.
5. Failure is defined as the final inability of the specimen to store elastic strain energy.
6. The amount of plastic work has no role in failure. It is not energy to be spent for something. It is failure by itself. The consequent exclusion of plastic strains cancels the distinction between brittle and ductile materials with respect to their failure.
7. It is desirable for failure criteria to be based on all components of stresses and strains. Otherwise, their validity is questionable.

In reality, plasticity plays a great role in the course of loading and different loading paths have as a result different amounts of plastic work stored in the material. This is evident through the continuous interaction between elastic moduli. But, at the "end of the day" the elastic energy (which is as it is, because of many factors including plastic strains) triggers failure. At the moment of failure elastic strain energy density is a characteristic quantity independent of loading path. So, the fundamental distinction between loading path and failure state (a single final point) must be clearly understood. Sometimes cross-purposes are created from the fact that failure is the last point of a loading path, which has as a result the connection of failure with plastic work. For a given material plastic strains are affected strongly by loading paths. In turn, plastic strains designate the exact expand of constitutive equations after elasticity. Imagine an ideal linear elastic—perfectly plastic material. It fails (nominally becomes a liquid) at the end of linearity. Beyond this point, all the externally supplied energy is converted to plastic work. Is it possible to locate a single strain point in the perfectly plastic branch of the constitutive equation as representing failure?

It is a rigid fact for almost one century ago that plastic strains show high sensitivity on stress system and/or specimen geometry [5,6]. This phenomenon is described by the introduction of various "stress triaxiality factors" and distinctions between plane stress and plane strain conditions. In practice, it implies that the amount of plastic strains takes quite different values at failure. A good and quite clear recent example is presented in Fig. 20 of the work by Bao and Wierzbicki [7], where, for cylindrical rods, equivalent plastic strain, $\overline{e}_f$, varies from 0.2 to 0.6 versus stress triaxiality. Then, which exactly is the critical value of $\overline{e}_f$ characterizing the failure of the specific material? Consequently, plastic strains (and plastic work) do not satisfy the fundamental requirement to keep a constant critical value at failure, characterizing the behavior of the material. This simple observation is strong enough to reject any failure criterion based on plasticity.

It is true that elastic strains depend on the same factors as the plastic ones, but in a much weaker quantitative manner. At a first glance one could say that, for the same as above reasons, elastic strains must also be excluded from the formulation of failure criteria. It is fully true when a single strain component is used. A Rankine type criterion with a single strain (or stress) critical quantity in the role of the critical one is, now, useless as common experience dictates. However, failure criteria based on combined stresses (as the original Mohr–Coulomb criterion) deserve a special merit, as it will be discussed in the sequel.

The obvious next step is the formulation of a criterion based on all stresses and all elastic strains i.e. based on elastic strain energy density. The question here is whether strain energy density (exactly speaking its two components) possesses the desired property of a unique for each material critical value at failure and this is our main task.

A launching point for the formation of such a criterion in case of linear elasticity was the first version of the so-called T-criterion which proved adequate to predict failure conditions for pre-cracked [8,9] or un-cracked geometries [10,11]. It was based on the von Mises [12] criterion and an addendum giving an answer to the question: "What happens with the dilatation of the material?" However, this version of the criterion was unable to give an answer to the phenomenon of "pressure dependence" of failure, exactly because it was based on linearity.

Pressure dependence of failure surfaces expresses an intrinsic property of materials as they *really* are, i.e. non-linear. Linear elasticity is a good tool for engineering applications, but it is not a general purpose tool. It is known that in terms of Thermodynamics, linearity is valid for all materials only at the first infinitesimal stress step. In terms of Mechanics, all brittle materials are non-linear elastic for the complete course of loading except a few initial loading steps. Ductile materials exhibit strong non-linear behavior at loads approaching the first yield point. Many attempts have been focused to this problem due to its high technological interest. Some of them [7,13–15] comply with theoretical requirements, the remaining [16–19] being more or less empirical and based on rich experimental evidence especially for brittle, non-linear elastic materials, like stones, marbles, etc.

A classical criterion clearly marking out pressure dependence is Coulomb [20] criterion, which after Mohr [21] states that a material fails when *"a proper combination of shear and normal stresses is realized"*. This statement is logically perfect (what else?) but lacks a quantitative description as far as it is based on experimental data and arbitrary assumptions for the shape of the failure envelope. Which one is the *proper combination*? Why it is used only in case of brittle materials? Does it imply that ductile materials fail when an *improper* combination of shear and normal stresses is realized? Of course not, Coulomb criterion simply reflects the only two geometrically permissible and discrete modes of failure activated during loading, i.e. slip caused by shear and cleavage caused by normal stresses. It interprets in terms of Mechanics the Euclidian perspective that: Any geometrical shape is formed by linear segments and angles. Or, in terms of Mechanics, a representative of the material elementary volume changes its volume and/or shape in order to store strain energy. *Tertium non datur*. This simple Euclidian truth is not reflected in all the as above mentioned criteria. In fact, both groups start from the classical expression $\tau = \alpha\sigma + \beta$ given by Mohr and evaluate experimentally the failure constants α and β. Theoretical criteria proceed to a second step by transforming normal and shear stresses σ and τ to other mechanical quantities (principal stresses, invariants, Lode angle, etc.) properly selected. Sometimes this second step drives to exaggerated situations.