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Exact solutions of sequential limit analysis of pressurized cylinders with combined hardening based on a generalized Hölder inequality: Formulation and validation

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ABSTRACT

The paper aims to investigate nonlinear combined isotropic/kinematic hardening cylinders under internal proportional pressure by sequential limit analysis. The Armstrong–Frederick kinematic hardening model is adopted and the Voce hardening law is incorporated for isotropic hardening behavior. In particular, we establish the kinematic formulation of sequential limit analysis from the corresponding static formulation by a generalized Hölder inequality. Especially, it is found that the derived kinematic formulation involving combined isotropic/kinematic hardening is equivalent to that by the bipotential concept. Further, exact solutions of plastic limit pressure were developed analytically by performing both static and kinematic limit analysis. Finally, the problem formulation and the solution derivations presented here are validated by a very good agreement between the numerical result of exact solutions of the present work and the upper bounds from the kinematic formulation available in literature.

1. Introduction

As it is well known, plastic limit load can be acquired directly by limit analysis using the static or the kinematic theorem [1]. In literature, investigation has been extensively made to plastic limit pressure of perfectly plastic structures (e.g. [2–4]). Further, the concept of sequential limit analysis has made it possible to deal with limit analysis problems involving hardening materials by updating the yield function and the deformed configuration during the deforming process [5]. It has been illustrated extensively [5–21] that sequential limit analysis is an accurate and efficient tool for the large deformation analysis. Further, a generalized Hölder inequality [22] has been utilized to establish the kinematic formulation of sequential limit analysis from the corresponding static formulation [5,8–21]. By a generalized Hölder inequality [22], we can theoretically confirm the equality relation between the greatest lower bound acquired by the static formulation and the least upper bound obtained by the kinematic formulation [5,8–21]. In the previous studies on effects of isotropic hardening [20,21], the first author analytically solved both static and kinematic limit analysis problems and acquired exact solutions for certain problems by confirming the equality relation between the greatest lower bound and the least upper bound.

On the other hand, it is noted that real-life materials generally demonstrate a combined isotropic/kinematic hardening behavior [23]. Therefore, it is interesting to extend the approach of sequential limit analysis to consider combined isotropic/kinematic hardening materials. Recently, Chaaba [24] applied the kinematic formulation of sequential limit analysis to seek the upper-bound limit pressure of thick vessels of combined isotropic/kinematic hardening based on the bipotential concept. By the bipotential concept, Chaaba [25] also developed both the static and kinematic formulations of sequential limit analysis involving combined isotropic/kinematic hardening materials. Accordingly, it is interesting to establish the static and kinematic formulations of sequential limit analysis to deal with combined hardening materials based on a generalized Hölder inequality [22]. In the paper, it is aimed to utilize a generalized Hölder inequality [22] to establish the kinematic formulation from the corresponding static formulation of a sequential limit analysis problem. Both static and kinematic limit analysis are to be analytically conducted sequentially to approach the real limit solutions. Particularly, the equality relation between the greatest lower bound and the least upper bound is to be confirmed explicitly to acquire the exact solution of limit pressure.

2. Analytical background

In the paper, we consider thick-walled hollow cylinders made of materials with nonlinear isotropic and kinematic hardening

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subjected to internal pressure in plane strain conditions. It is aimed to apply the concept of sequential limit analysis to investigate plastic limit pressure of the axis-symmetric plane strain problem. The behavior of nonlinear isotropic and kinematic hardening is described by the Voce hardening law [26] and the Armstrong–Frederick kinematic hardening model [27], respectively. Corresponding to the nonlinear combined isotropic/kinematic hardening for a von Mises material, the yield function is denoted as [28]

$$f(\sigma - X) = \sqrt{\tfrac{3}{2}(S - X^{dev}) : (S - X^{dev})} - \sigma_Y \tag{1}$$

where S is the deviatoric stress tensor, X^{dev} is the deviatoric part of the backstress tensor X acting to translate the center of the yield surface, σ_Y is the yield strength. The yield strength σ_Y is the isotropic hardening parameter accounting for the change in size of the yield surface. On the other hand, the backstress X is the kinematic hardening parameter resulting in the movement of the center of the yield surface. Accordingly, the yield function can be completely updated, in terms of the size and the shape, in sequential limit analysis once the yield strength σ_Y and the backstress tensor X are prescribed in each step. Moreover, the shape of the yield surface is unchanged while the combined isotropic/kinematic hardening model is considered. Therefore, it is expected that the convexity of the yield function is reserved.

By the Armstrong–Frederick kinematic hardening model [27], the backstress rate $\dot{X}$ is described as

$$\dot{X} = \tfrac{2}{3} C \dot{\varepsilon} - \gamma X \dot{\bar{\varepsilon}} \tag{2}$$

where C and γ are the material parameters, $\dot{\varepsilon}$ is the plastic strain rate, $\dot{\bar{\varepsilon}}$ denotes the equivalent strain rate.

The behavior of nonlinear isotropic hardening is modeled by the Voce hardening law [26] in the following form:

$$\sigma_Y = \sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon}) \tag{3}$$

where σ_0 is the initial yield strength, σ_∞ is the saturation value of σ_0, h is the hardening exponent and $\bar{\varepsilon}$ is the equivalent strain.

We consider a thick-walled cylindrical vessel with the initial interior and exterior radii indicated by a_0 and b_0, respectively. After the action of internal pressure, the current interior and exterior radii are denoted by a and b in the induced widening process, respectively. In addition, the cylindrical vessel subjected to internal pressure assumes the boundary conditions $\sigma_r(r=a) = P_i$ and $\sigma_r(r=b) = 0$ with P_i the value of internal pressure and r the current radius, respectively.

Further, corresponding to the von Mises yield criterion with the associated flow rule, we have the equivalent stress $\bar{\sigma}$ and the equivalent strain rate $\dot{\bar{\varepsilon}}$ as follows:

$$\bar{\sigma} = \sqrt{\tfrac{1}{2}[(\sigma_\theta - X_\theta) - (\sigma_z - X_z)]^2 + \tfrac{1}{2}[(\sigma_z - X_z) - (\sigma_r - X_r)]^2 + \tfrac{1}{2}[(\sigma_r - X_r) - (\sigma_\theta - X_\theta)]^2} \tag{4}$$

$$\dot{\bar{\varepsilon}} = \sqrt{\tfrac{2}{9}\left[(\dot{\varepsilon}_\theta - \dot{\varepsilon}_z)^2 + (\dot{\varepsilon}_z - \dot{\varepsilon}_r)^2 + (\dot{\varepsilon}_r - \dot{\varepsilon}_\theta)^2\right]} \tag{5}$$

where σ_r, σ_θ and σ_z are the stress components in the radial, circumferential and axial directions, respectively. X_r, X_θ and X_z are the backstress components in the radial, circumferential and axial directions, respectively. $\dot{\varepsilon}_r$, $\dot{\varepsilon}_\theta$ and $\dot{\varepsilon}_z$ are the strain rates components in the radial, circumferential and axial directions, respectively.

Considering plane strain conditions and the incompressibility, we can further reduce the equivalent stress $\bar{\sigma}$ and the equivalent strain rate $\dot{\bar{\varepsilon}}$ into the following simplified forms:

$$\bar{\sigma} = \tfrac{\sqrt{3}}{2}[-(\sigma_r - X_r) + (\sigma_\theta - X_\theta)] \tag{6}$$

$$\dot{\bar{\varepsilon}} = -\tfrac{2}{\sqrt{3}}\dot{\varepsilon}_r = \tfrac{2}{\sqrt{3}}\dot{\varepsilon}_\theta \tag{7}$$

On the other hand, due to the axis-symmetry, the strain rate–velocity relations are given by

$$\dot{\varepsilon}_r = \frac{du_r}{dr} \tag{8}$$

$$\dot{\varepsilon}_\theta = \frac{u_r}{r} \tag{9}$$

where u_r is the velocity component in the radial direction. Also, we have the incompressibility in plane strain conditions described in the form

$$\nabla \cdot \vec{u} = \dot{\varepsilon}_r + \dot{\varepsilon}_\theta = 0 \tag{10}$$

Accordingly, we have the radial velocity indicated in the form

$$u_r = \frac{a\dot{a}}{r} \tag{11}$$

where $\dot{a}$ is the velocity of the current interior radius.

Accordingly, we rewrite the equivalent strain rate $\dot{\bar{\varepsilon}}$ in the form

$$\dot{\bar{\varepsilon}} = \frac{2}{\sqrt{3}}\frac{a\dot{a}}{r^2} \tag{12}$$

The equivalent strain is then obtained as

$$\bar{\varepsilon} = \int \dot{\bar{\varepsilon}} dt = \frac{1}{\sqrt{3}} \ln \frac{r^2}{r_0^2} \tag{13}$$

where r_0 is the initial radius of the location concerned.

Combining Eqs. (3) and (13), the Voce hardening law [26] can be rewritten in the form as

$$\sigma_Y = \sigma_\infty - (\sigma_\infty - \sigma_0)\left(\frac{r_0}{r}\right)^{2h/\sqrt{3}} \tag{14}$$

Also, we can obtain the backstress X by solving Eq. (2) with the initial condition $X(0)$. The components of the backstress rate $\dot{X}$ can be described as follows by combining Eqs. (2) and (7):

$$\dot{X}_r = \tfrac{2}{3} C \dot{\varepsilon}_r + \gamma X_r \tfrac{2}{\sqrt{3}} \dot{\varepsilon}_r \tag{15}$$

$$\dot{X}_\theta = \tfrac{2}{3} C \dot{\varepsilon}_\theta - \gamma X_\theta \tfrac{2}{\sqrt{3}} \dot{\varepsilon}_\theta \tag{16}$$

Considering the proportional loading and the initial condition $X(0) = 0$, the integral form of Armstrong–Frederick kinematic hardening model [27] can be obtained as follows:

$$X_r = -\frac{C}{\sqrt{3}\gamma} + \frac{C}{\sqrt{3}\gamma} e^{(2\gamma/\sqrt{3})\varepsilon_r} = -\frac{C}{\sqrt{3}\gamma} + \frac{C}{\sqrt{3}\gamma}\left(\frac{r_0}{r}\right)^{2\gamma/\sqrt{3}} \tag{17}$$

$$X_\theta = \frac{C}{\sqrt{3}\gamma} - \frac{C}{\sqrt{3}\gamma} e^{-(2\gamma/\sqrt{3})\varepsilon_\theta} = \frac{C}{\sqrt{3}\gamma} - \frac{C}{\sqrt{3}\gamma}\left(\frac{r_0}{r}\right)^{2\gamma/\sqrt{3}} \tag{18}$$

Note that, the condition that $\gamma \neq 0$ is assumed in deriving Eqs. (17) and (18).

By Eqs. (14), (17) and (18), respectively, we can obtain the yield strength σ_Y and the backstress X as functions of the location r. Accordingly, the yield function is completely updated with the step-wise constants of the yield strength σ_Y and the backstress X for sequential limit analysis involving combined isotropic/kinematic hardening materials.

3. Static and kinematic limit analysis

As shown in the previous section, the yield strength σ_Y and the backstress X can be considered as step-wise constants for a given configuration. Therefore, we can apply the concept of sequential limit analysis to deal with combined isotropic/kinematic hardening by updating the yield function and the deformed configuration.

In the following sections, analytical efforts of both static and kinematic limit analysis are to be made. On the one hand, we seek