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Wave dispersion and attenuation in viscoelastic polymeric bars: Analysing the effect of lateral inertia

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A B S T R A C T

Elementary one-dimensional wave theory is often used to describe the propagation of longitudinal stress waves in polymer rods. More accurate solutions are available, but they are mathematically difficult. A new wave equation is derived for long polymeric rods in this paper. The material properties are modelled as a Maxwell viscoelastic material acting in parallel with an elastic material. Lateral motions of the rod that result from the Poisson effect are accounted for using a new concept called the "effective density". The effects of both the material properties and the diameter of the bar on dispersion and attenuation coefficients are highlighted. The new wave theory simplifies to the one-dimensional solution for waves in polymer rods if the Poisson ratio is set to zero. The predictions simplify to Love's equation for stress waves in elastic bars when rate dependency is removed from the material model.

1. Introduction

The dynamic testing of materials and components often involves predicting the propagation of stress waves in slender rods. For example, the Split Hopkinson Pressure Bar (SHPB) is a well known apparatus for determining the mechanical properties of materials under intermediate rate loading. The forces and displacements at the two bar–specimen interfaces are determined by strain measurements at locations along the bars. When testing soft materials (foams, polymers) using a conventional SHPB set-up with metallic bars, the majority of the incident pulse is reflected back and only a small portion of the incident wave passes through the specimen to the transmitter bar. This leads to difficulties in sensing the transmitter signal with sufficient accuracy. A greater problem is that the incident and the reflected waves will be almost equal in magnitude. It then becomes extremely difficult to verify equilibrium in the specimen. Gray and Blumenthal [1] give a detailed account of the techniques employed during SHPB testing of soft materials. Finite element modelling of the whole Hopkinson bar system (including the sample) is suggested in order to support the quantification of the constitutive behaviour of materials. Polymer pressure bars have been employed to overcome the difficulties associated with SHPB testing of soft materials. The propagation of stress waves along these bars using a linear viscoelastic rheological model is the focus of this paper. The reader is referred to Ref. [1] for descriptions of sample behaviour and to Ref. [2] for models that incorporate nonlinear mechanical properties of polymers as a function of temperature, strain and strain rate.

Since polymer bars are viscoelastic, the longitudinal stress waves attenuate when propagating through the rod (see, e.g. [3]). Both the attenuation coefficient and the phase velocity of polymeric bars are frequency dependent and finding an appropriate function to express this dependency is vital in determining the displacements and stresses at the ends of the bars. Some researchers have employed advanced processing techniques [4] and intelligent computational algorithms [5] to account for the attenuation and dispersion effects. Other studies have concentrated on wave theories. Generally, wave propagation has been analysed in the frequency domain. The analysis often involves a complex Young's modulus combined with, e.g. one-dimensional wave equation (see, e.g. [3]). For greater accuracy at higher frequencies, Zhao and Gary [6] generalised the Pochhammer–Chree frequency equation for elastic bars to viscoelastic bars, where the elastic constants were replaced by complex properties. This work has subsequently been employed in SHPB studies (e.g. Ref. [7]). However, Zhao and Gary [6] adopted a mathematically complicated approach wherein a nine-parameter rheological model accounted for both dispersion and attenuation effects by generalising the Pochhammer–Chree equation for elastic bars to viscoelastic bars, while assuming that the Poisson ratio was constant. The level of complexity inherent to the analysis in [6] may be the reason that few other groups have attempted to repeat the work. The difficulties associated with the technique are discussed in [8], where approximations to the Pochhammer–Chree equation for viscoelastic bars are employed to extend the usable frequency range beyond that for the one-dimensional theory. The majority of studies on

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longitudinal stress waves in polymer rods retain the one-dimensional wave theory for simplicity, e.g. see [9–12].

Time-domain analysis of wave propagation in polymer rods has received less attention. However, the viscoelastic effect has been incorporated in the one-dimensional wave equation [13], which is valid within a certain frequency range. Experimentally, this range can prove limiting. For elastic bars, the effect of lateral inertia was predicted by Love [14] a century ago. Love defined the kinetic and strain energy in an elastic rod and derived an equation that incorporated the effect of lateral motion on axial waves using the calculus of variations (see [14,15]). It is not possible to repeat this analysis for a viscoelastic material as the necessary expressions for strain energy are not derivable. An alternative derivation is presented here. Due to the Poisson effect, there is kinetic energy in lateral as well as axial motion. This kinetic energy is used to define a new concept called the "effective density", which is related to the density via a differential operator. Thereafter, expressing Newton's second law in terms of the "effective density" results in a fifth order partial differential equation (PDE), which represents longitudinal waves in a viscoelastic bar. The new predictions highlight the effect of bar diameter on both attenuation and dispersion coefficients of epoxy rods.

2. Derivation of the new wave equation

In the longitudinal (x direction) the strain, strain rate and particle velocity associated with the displacement field u are

$$\varepsilon = \frac{\partial u}{\partial x}, \tag{1}$$

$$\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial t} = \frac{\partial^2 u}{\partial x \, \partial t}, \tag{2}$$

$$V = \dot{u} = \frac{\partial u}{\partial t}, \tag{3}$$

respectively. Due to the Poisson ratio υ, there are displacement fields v and w in the y and z directions, respectively. For example, the strain and the derivative of the displacement with time in the y direction are

$$\varepsilon_y = -\upsilon \varepsilon, \tag{4}$$

$$\dot{v} = -\upsilon y \dot{\varepsilon}, \tag{5}$$

respectively. The kinetic energy of an infinitesimal length Δx of the bar (Fig. 1) is

$$\Delta T_e = (V^2 + \upsilon^2 k^2 \dot{\varepsilon}^2)\rho A \frac{\Delta x}{2}, \tag{6}$$

where ρ is density, A is cross-sectional area and k is radius of gyration of the solid circular cross-section.

Zhang and Yu [16] analysed the dynamic compression of an inertia-sensitive energy-absorbing structure. The structure consisted of two plates that were pre-bent to a small angle and then fastened together at the top and base. The plates were then compressed axially by a falling rigid mass. The deformation of the plates was assumed to consist of the rotation of four rigid bars that were connected by plastic hinges. Lagrange's equation of the second kind was used to derive an equation of motion of the system. The behaviour of the system is analogous to that of a completely plastic in-line collision of two unequal masses, i.e. the rigid falling mass and the "effective mass" of the two plates. The effective mass of the plates consists of two terms. These two terms represent the "longitudinal" and "transverse" inertia of the actual specimen. Here, an "effective density" is used to incorporate the effect of lateral inertia in a one-dimensional wave equation. The mathematical rigour of the derivation of the effective mass in [16] is not attempted. Rather, it is postulated that the effect of lateral inertia effect can be incorporated in a wave model using the concept of "effective density". The validity of the approach is supported by comparison with two other wave theories. It is then shown that the new wave equation is equivalent to Love's equation for stress waves in elastic bars when rate dependency is removed from the material model. Furthermore, the new equation simplifies to the one-dimensional solution derived by Wang et al. [13] for waves in viscoelastic polymer rods if the Poisson ratio is set to zero.

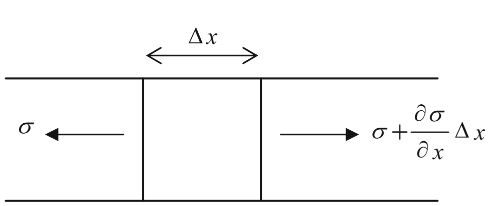

Fig. 1. Longitudinal forces on element of bar.

An effective density ρ_e is now introduced such that the kinetic energy of the element Δx is ΔT_{ea}, where

$$\Delta T_{ea} = V^2 \rho_e A \frac{\Delta x}{2}. \tag{7}$$

Using Newton's second law, the net longitudinal force acting on the element Δx (Fig. 1) is

$$A \Delta \sigma = \left(\rho_e A \frac{\partial^2 u}{\partial t^2} \right) \Delta x. \tag{8}$$

As $\Delta x \to 0$, Eq. (8) can be written as

$$\frac{\partial \sigma}{\partial x} = \rho_e \frac{\partial^2 u}{\partial t^2}. \tag{9}$$

To find the relationship between ρ_e and ρ, a functional T_D is defined as the kinetic energy error when using the effective density ρ_e, i.e.

$$T_D = \iint \left[\left(0.5 A \rho V^2 + 0.5 A \rho \upsilon^2 \kappa^2 \dot{\varepsilon}^2 \right) - \left(0.5 A \rho_e V^2 \right) \right] dx\, dt. \tag{10}$$

Substituting for strain rate (Eq. (2)) and particle velocity (Eq. (3)) in Eq. (10), the minimisation of T_D requires minimisation of the following functional:

$$I = \iint \left[\left(\rho V^2 + \rho \upsilon^2 \kappa^2 \left(\frac{\partial V}{\partial x} \right)^2 \right) - (\rho_e V^2) \right] dx\, dt. \tag{11}$$

The functional in Eq. (11) can be written in terms of displacement u as

$$I = \iint \left[(\rho - \rho_e) \left(\frac{\partial u}{\partial t} \right)^2 + \rho \upsilon^2 \kappa^2 \left(\frac{\partial^2 u}{\partial x \, \partial t} \right)^2 \right] dx\, dt. \tag{12}$$

The integrand in the functional I is

$$f\left(\frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x \, \partial t} \right) = (\rho - \rho_e) \left(\frac{\partial u}{\partial t} \right)^2 + \rho \upsilon^2 \kappa^2 \left(\frac{\partial^2 u}{\partial x \, \partial t} \right)^2. \tag{13}$$

According to Akhiezer [17], in order to minimise I in Eq. (12), the integrand f in Eq. (13) should satisfy the following equation:

$$\frac{\partial^2}{\partial x \, \partial t} \left(\frac{\partial f}{\partial u_{xt}} \right) - \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial u_t} \right) = 0. \tag{14}$$

In Eq. (14) $u_{xt} = (\partial^2 u / \partial x \, \partial t)$ and $u_t = (\partial u / \partial t)$. Therefore Eq. (14) is identical to the following partial differential equation:

$$\rho_e \frac{\partial^2 u}{\partial t^2} = \rho \frac{\partial^2 u}{\partial t^2} - \rho \upsilon^2 k^2 \frac{\partial^4 u}{\partial x^2 \, \partial t^2}. \tag{15}$$

Eq. (15) gives the relationship between effective density and density that minimises the effective density error, T_D, defined in