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Large amplitude free vibration analysis of Timoshenko beams using a relatively simple finite element formulation

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ABSTRACT

Free vibration analysis of uniform isotropic Timoshenko beams with geometric nonlinearity is investigated through a relatively simple finite element formulation, applicable to homogenous cubic nonlinear temporal equation (homogenous Duffing equation). Geometric nonlinearity is considered using von-Karman strain displacement relations. The finite element formulation begins with the assumption of the simple harmonic motion and is subsequently corrected using the harmonic balance method. Empirical formulas for the non-linear to linear radian frequency ratios, for the boundary conditions considered, are presented using the least square fit from the solutions of the same obtained for various central amplitude ratios. Numerical results using the empirical formulas compare very well with the results available from the literature for the classical boundary conditions such as the hinged–hinged, clamped–clamped and clamped–hinged beams. Numerical results for the beams with non-classical boundary conditions such as the hinged-guided and clamped-guided, hitherto not studied, are also presented.

1. Introduction

Slender and short beams are widely used structural members in aerospace structures. Accurate evaluation of the dynamic characteristics such as frequencies and modeshapes of the structures subjected to large amplitude vibration essentially helps in better understanding the performance of the structure.

Large amplitude free vibration analysis of beams with geometric non-linearity has been investigated by various researchers [1–26] using either the analytical or the approximate continuum and numerical methods since the classic work of Woinowsky-Krieger [1].

Energy method involves in assuming suitable admissible displacement functions for the lateral displacement, total rotation and also axial displacement if included in the formulation. This leads to two nonlinear temporal differential equations in terms of the lateral displacement and total rotation if one excludes the axial displacement in the formulation of large amplitude vibration of Timoshenko beams and are difficult to solve to obtain the large amplitude frequencies as a function of the amplitude and slenderness ratios of the vibrating beam. Recent references related to coupled displacement field method for Timoshenko beams can be seen in the Refs. [22–25]. Researchers [22–25] used the coupled displacement field method in which the one displacement variable is coupled to the other using the static equilibrium equation of the shear flexible beam so that complexity of the problem gets reduced. If one includes the axial displacement also in the formulation apart from lateral displacement and total rotation, the complexity of the problem increases and they are not easily amenable to solve the problem of large amplitude vibration of Timoshenko beams using the energy method. It also increases the complexity of the problem in selecting a proper admissible function for the complicated boundary conditions in which many displacement variables are associated and it leads to coupled differential equations.

In the present study, the large amplitude free vibration analysis of the uniform Timoshenko beams considering the effects of transverse shear and rotary inertia is investigated with all possible boundary conditions, where the ends of the beam constrained to move axially, resulting in von-Karman type strain–displacement relation. A comprehensive study is carried out for the hinged–hinged (H–H), clamped–clamped (C–C), clamped–hinged (C–H), clamped–guided (C–G) and hinged–guided (H–G) beams starting with the SHM assumption. The final solution in terms of ratios of the nonlinear to linear radian frequencies for several central amplitude ratios is obtained by applying the harmonic balance method (HBM) [16] to correct the error involved in the assumption of the SHM for the above-mentioned boundary conditions. The guided boundary condition considered is of two types and are denoted by G1 and G2, where G1 represents that the lateral displacement and the total rotation are

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Nomenclature

a	central amplitude of the vibrating beam
E	Young's modulus
G	Shear modulus ($E/2(1+\nu)$)
I	area moment of inertia
κ	shear correction factor
$[K_e]$	element stiffness matrix
$[K]$	assembled stiffness matrix
l	element length
L	length of the beam
m	mass per unit length
$[M_e]$	element mass matrix
$[M]$	assembled mass matrix
r	radius of gyration
u	axial displacement
U	strain energy
w	lateral displacement
T	kinetic energy
x	axial coordinate
$\alpha_1 \to \alpha_{12}$	generalised co-ordinates
ε_x	axial strain
λ	eigenvalue (frequency parameter $= m\omega^2 L^4/EI$)
ψ_x	curvature
ω	radian frequency
δ	eigenvector (mode shape of vibration)

Subscripts

L	linear
NL	nonlinear
H	harmonic

Superscript

$()'$	differentiation with respect to x
$[]^T$	transpose of a matrix

not constrained and in G2, there is no constraint to lateral displacement while the total rotation is constrained. It may be emphasized that the matrices involved in the eigenvalue problem are symmetric in the present FE formulation. Numerical results for the classical boundary conditions H–H, C–C and C–H beams are available in the literature and the present results compare very well with those and at the same time the corresponding results for the non-classical boundary conditions C–G1, C–G2, H–G1 and H–G2 beams are not readily available and are presented perhaps for the first time. The simplicity of the present FE formulation lies in getting the realistic solution using the HBM [16] to correct for the assumption of the SHM contrary to the procedures followed in Refs. [13,14].

2. Finite element formulation

Let the beam is divided into a number of uniform finite elements of length l. The strain energy (U) of the element is given by

$$U = \frac{EA}{2}\int_0^l \varepsilon_x^2\,dx + \frac{EI}{2}\int_0^l \left(\frac{d\theta}{dx}\right)^2 dx + \frac{\kappa GA}{2}\int_0^l \left(\frac{dw}{dx}+\theta\right)^2 dx \tag{1}$$

where the non-linear strain–displacement relation of the beam with the axially immovable ends (von-Karman type) are given by

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{2}$$

Note that Eq. (2) is valid for small strain but moderately large rotation and transverse deflection (of the order of characteristic dimension of the cross-section) of the beam [27,28].

Using Eqs. (1)–(2) the element nonlinear elastic stiffness matrix $[K_e]_{NL}$ interms of generalized coordinate θ, $d\theta/dx$, du/dx and dw/dx in the symmetric form is

$$[K_e]_{NL} = \frac{EI}{2}\int_0^l \begin{bmatrix} \frac{\kappa}{(1+\nu)r^2} & 0 & 0 & \frac{\kappa}{(1+\nu)r^2} \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{2}{r^2} & \frac{w'}{r^2} \\ \frac{\kappa}{(1+\nu)r^2} & 0 & \frac{w'}{r^2} & \frac{u'}{r^2}+\frac{(w')^2}{r^2}+\frac{\kappa}{(1+\nu)r^2} \end{bmatrix} dx \tag{3}$$

The kinetic energy of the vibrating element including the effect of rotary inertia, neglecting axial inertia followed by most of the researchers, with the assumption of the SHM is given by

$$T = \frac{1}{2}\int_0^l m\dot{w}^2\,dx + \frac{1}{2}\int_0^l I\dot{\theta}^2\,dx = \frac{m\omega^2}{2}\int_0^l w^2\,dx + \frac{I\omega^2}{2}\int_0^l \theta^2\,dx \tag{4}$$

Cubic polynomial distributions are assumed over the beam element for θ, u and w in terms of the axial coordinate x as

$$\theta = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 \tag{5}$$

$$u = \alpha_5 + \alpha_6 x + \alpha_7 x^2 + \alpha_8 x^3 \tag{6}$$

$$w = \alpha_9 + \alpha_{10} x + \alpha_{11} x^2 + \alpha_{12} x^3 \tag{7}$$

It is to be noted that the assumption of lower order polynomials for θ and w in finite element (FE) formulation results in shear locking effect [27] and also assuming a lower order polynomials for u and w results in membrane locking phenomenon in beams due to the nature of coupled displacement fields in the FE procedure [27]. However, here in the present work cubic polynomials are chosen for θ, u and w.

The nonlinear elastic stiffness $[K_e]_{NL}$ and mass $[M_e]$ matrices are obtained in terms of the nodal parameters θ, $d\theta/dx$, u, du/dx, w, dw/dx using the standard congruent transformations, where the two transformation matrices $[\mathbf{T_1}]$ and $[\mathbf{T_2}]$ are given by

$$\begin{bmatrix} \theta \\ \frac{d\theta}{dx} \\ u \\ \frac{du}{dx} \\ w \\ \frac{dw}{dx} \end{bmatrix} = [\mathbf{T}_1][\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5\ \alpha_6\ \alpha_7\ \alpha_8\ \alpha_9\ \alpha_{10}\ \alpha_{11}\ \alpha_{12}]^T \tag{8}$$

$$[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ .\ .\ \alpha_{11}\ \alpha_{12}]^T = [\mathbf{T}_2][\theta_1\ \theta'_1\ u_1\ u'_1\ w_1\ w'_1\ \theta_2\ \theta'_2\ u_2\ u'_2\ w_2\ w'_2]^T \tag{9}$$

where the subscripts 1 and 2 in the nodal physical displacement vector represent the nodes 1 and 2 of the beam element, respectively.

After the assembly of the element matrices, the governing matrix equation for the large amplitude free vibration phenomenon with the assumption of the SHM is

$$[K]_{NL}\{\delta\} - \lambda_{NL}[M]\{\delta\} = 0 \tag{10}$$