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The applicability of the effective medium theory to the dynamics of cellular beams

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ABSTRACT

The applicability and the limitations of the effective medium assumption for the dynamics of cellular beams are studied. Beams made of uniform triangular and regular hexagonal cells are analysed. The natural frequencies and modal distributions as calculated using the detailed finite element model of the cellular networks are compared with those predicted based on equivalent homogeneous media of the same overall size and shape. It is observed that, for low mode number, a cellular beam behaves as a continuum, provided the cell size is significantly smaller than the external dimensions of the beam. Due to different deformation mechanisms, triangular cells show frequencies independent of area fraction whereas hexagonal cells show this dependence clearly. As the wavelength starts to become of the order of the heterogeneity, the continuum behaviour begins to break down. With the increase in mode number, cellular beams exhibit inherent flexibility with a progressive increase in their modal densities as compared to those of a homogeneous continuum. The modal density increases further when the cell walls start to resonate. During resonance, an abrupt rise in the modal density is observed for the triangular cells as the cell walls start deforming in the flexural mode instead of the axial mode. In contrast, for hexagonal cells, the predominant mode of cell wall deformation is always flexural.

1. Introduction

Natural and synthetic cellular materials such as wood, cork, bone, polymeric, ceramic and metallic foams have inspired many theoretical, experimental and computational works relating structure to properties. Due to their high stiffness to weight ratio, they are increasingly being used for various industrial applications. Honeycombs and foams are used as core of sandwich structures in aerospace and marine industry. Industrial applications include packaging, thermal insulation, acoustic damping, biomedical scaffolds, etc.

At the mesoscopic scale, i.e. at the length scale of a typical cell, cellular materials are made of complex network of elastic beams or plates. When the overall size of a piece of cellular material is much larger than the length scale of a typical cell, the structure can be treated as homogeneous with some effective properties. The literature shows that the research on cellular structures is primarily devoted in calculating its effective properties, given the geometry, topology, and material properties of the cell wall material.

A generic approach used to analyse periodic cellular structures is to identify a suitable unit cell as a representative of the whole, and then analyse the mechanics of the unit cell for predicting the macroscopic behaviour. A few analytical works are available so far using this sort of *unit cell approach* [1–3]. For example, Gibson et al. [1] determined the in-plane elastic constants for infinite lattice made of hexagonal cells. In their unit cell approach, they considered only the bending deformation of the cell walls. Warren and Kraynik [2] used kinematic arguments to analyse the deformations of the cell walls connected at a node. They included both the axial and the bending deformations of the cell walls and evaluated the elastic constants for hexagonal honeycombs for all densities. Gulati [3] derived the expressions for Young's modulus for honeycombs made of triangular cells using strain energy based method.

In case of the *numerical approach*, the unit cell is modelled as a network of interconnected beams. The bulk behaviour is predicted by analysing the unit cell using the finite element method [4–9]. For instance, Scarpa et al. [4] used experimental techniques as well as the finite element method to analyse the in-plane elastic behaviour of honeycombs made of inverted hexagons due to uniaxial loading. Silva et al. [5] and Zhu et al. [6] studied the effect of the microstructural variability on the elastic behaviour of two-dimensional cellular solids using the finite element method.

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Because of the advantage of modelling the intricate geometrical features of the microstructure, finite element method has been used for modelling various defects. For example, Silva et al. [7] analysed the effect of the local defects (e.g. removal of the cell walls) in an irregular topology on the bulk compressive failure behaviour. Grenestedt [8] studied the effect of the wavy nature of the cell walls. Simone and Gibson [9] analysed the dependence of the elastic properties on the thickness variations of the cell walls. Chen et al. [10] studied how these imperfections can affect the yielding of a two-dimensional foam.

Homogenisation theory has been used for calculating the bounds on the bulk properties of the cellular materials. Torquato et al. [11] derived bounds on the effective properties based on the Hashin–Shtrikman bounds and compared their results with the finite element predictions. An extensive discussions on the mechanical behaviour of cellular materials such as elastic, plastic, buckling, failure, thermal conductivity, etc. can be found in the work of Gibson and Ashby [12]. Christensen's review [13] describes the relationships between the elastic properties, cellular topology and the effective density for two- and three-dimensional materials. Mechanical behaviour of metallic foams can be found in Gibson's work [14]. Grenestedt [15] analysed various models for studying the mechanics of perfect cellular materials.

The literature shows that most of the research on cellular materials is devoted to the study of elastic and plastic behaviour under static loads, whereas very few works are available on the dynamics of such materials. Wang and Stronge [16] used a micropolar theory to analyse the behaviour of hexagonal honeycombs under periodic forces. Baker et al. [17] studied the effect of impact and energy absorption behaviour of cellular materials.

The present authors [18,19] developed a numerical scheme for reducing the computational expense associated with the free vibration and the response calculations of the cellular structures. As opposed to this, here we investigate the applicability and the limitations of the effective medium theory for dynamics of cellular structures. The effective properties based on the statics of infinite lattices are expected to predict the long wave behaviour well. However, it is not clear (1) what happens if a structure is not of infinite extent, because such structure is not strictly periodic, (2) how the dynamic behaviour changes with the increase in the mode number. The first question, in fact, raises the issue of the size of the specimen relative to the cell. In statics, researchers have observed softening effect during compression and stiffening effect in shear when the foam specimen size is not sufficiently large in comparison to the cell size [20,21]. It is not clear how these effects translate in case of dynamics and what their dependence on mode number is.

Cellular structures are modelled here using the finite element method. This allows modelling of all the geometric features of the structure in detail. In addition, this micromechanics-based approach does not impose any restriction on the size of the model or the nature of the boundary conditions. Periodic boundary conditions are not required in this case as the whole structure itself is analysed. The results from the numerical experiments are chosen as the benchmark for comparisons with the predictions based on the equivalent medium theory. Trends observed are explained using physical reasoning qualitatively and quantitatively.

The paper is organised as follows. An effective medium theory for the dynamics of cellular beams is presented in Section 2, followed by the results and discussions for the low frequency vibration of cellular beams in Section 3. Progressive deviation in the continuum behaviour with increase in mode number and the statistics of modal distribution are described in Section 4. Finally, the concluding remarks are made in Section 5.

2. Dynamics of cellular beams as described by effective medium

2.1. Finite element modelling of an elastic network

Free vibration of structures made of uniform triangular and regular hexagonal cells are analysed first. These cell topologies represent two extreme cases: the node connectivity of the cell walls is three for hexagonal cells and six for triangular cells. The dominant strain energy mechanism for the cell walls in a triangular topology is axial stretching, whereas for hexagonal cells bending of the cell walls dominates. Such lattices are isotropic in the plane. Our choice was also guided by the fact that the analytical expressions of the elastic constants for lattices made up of such cells are available in the literature.

Cell walls are modelled as two node or three node Euler–Bernoulli beams having three degrees-of-freedom at each node. The width normal to the plane of the structure is taken as unity. The degrees-of-freedom transverse to a cell wall along with the rotation at the end nodes account for the beam bending energy of the cell walls. The degrees-of-freedom in the direction of the line joining the end nodes of a cell wall account for the stretching energy. Hence, the displacements in the horizontal and the vertical directions along with the rotation of all the nodes approximately describe the deformed configuration of the whole structure. Within each beam element, cubic interpolation is used for the transverse displacements, whereas linear interpolation is used for the axial displacements. The details of interpolation and development of the global discretised equations are omitted here for brevity and can be found in standard textbooks on the finite element method (see, e.g. [22]). The essence of the procedure is described in the next paragraph. The shear effects in the cell walls have been ignored due to thinness and because the deformed shape of each cell wall does not show rapid fluctuations for low frequency dynamics. It is assumed that all the cell walls are of uniform thickness with zero damping. Displacements are assumed to be small so that a linear analysis is valid. The rotary inertia of the cell walls is neglected.

The overall strain energy V and the kinetic energy T can be expressed in terms of the generalised co-ordinates of the structure as

$$V = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}, \quad T = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}}, \tag{1}$$

where $\mathbf{q}$ is the vector of the generalised co-ordinates, $\mathbf{K}$ and $\mathbf{M}$ are the global stiffness matrix and global inertia matrix, respectively, of size $N \times N$ for a structure having N degrees-of-freedom. For a conservative system, the Lagrangian L takes the form $L = T - V$. Applying Hamilton's principle $\delta \int_{t_1}^{t_2} L\,dt = 0$ for a conservative system between the time interval t_1 and t_2, (δ means 'variation of'), we obtain the set of equations of motion for the undamped free vibration of cellular structure

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}. \tag{2}$$

For synchronous free vibration, Eq. (2) leads to the well known algebraic eigenvalue problem

$$\mathbf{K}\mathbf{u}_r = \lambda_r\mathbf{M}\mathbf{u}_r, \tag{3}$$

where λ_r is the r-th eigenvalue and, $\mathbf{u}_r$ is the corresponding eigenvector. The square root of the eigenvalue λ_r is the r-th natural frequency ω_r of the overall structure. Eigenvectors are the mode shapes of the structure.

Structures made of cells possess geometrical features at two scales of length (see, Fig. 1). First, the length scale of the overall structure, say L, is typically represented by the characteristic length of a hypothetical solid when the porosity is completely filled. There is another length scale intrinsic to the microstructure,