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# A unified generalized thermoelasticity; solution for cylinders and spheres

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**Abstract**

A new unified formulation for the generalized theories of the coupled thermoelasticity based on the Lord–Shulman, Green–Lindsay, and Green–Naghdi models is proposed in this paper. The unified form of the governing equations is presented by introducing the unifier parameters. The formulations are derived and given for the anisotropic heterogeneous materials. The unified equations are reduced for the isotropic and homogeneous materials. Transforming the governing equations into the Laplace domain, they are analytically solved in the space domain for a hollow sphere and cylinder, where a parameter is introduced to consolidate the solution for the sphere and cylinder in a unified form. A thermal shock load is applied to the inner surface of the sphere and cylinder and the results are presented using a numerical inversion technique of the Laplace transform. The results are validated with the known data in the literature.


*Keywords:* Generalized thermoelasticity; Unified formulation; Thick cylinders; Thick spheres; Thermal shock

## 1. Introduction

The conventional theory of thermoelasticity is based on the Fourier's heat conduction law. Due to the parabolic nature of the energy equation of this theory, infinite propagation speeds for the thermal disturbances are predicted. The concept of the hyperbolic nature involving finite speeds of thermal disturbance is reported by Maxwell [1] for the first time, known as the second sound. Chester [2] provides some justification to the fact that the so-called second sound must exist in any solid. Most of the approaches that came out to overcome the unacceptable prediction of the classical theory are based on the general notion of relaxing the heat flux in the classical Fourier heat conduction equation, thereby introducing a non-Fourier effect. One of the simplest forms of these equation is due to the work of Lord and Shulman [3]. In the Lord–Shulman (LS) theory a relaxation time is introduced and the Fourier's heat conduction equation is modified. Another thermoelasticity theory that admits the second sound effect is reported by Green and Lindsay [4]. In the Green–-Lindsay (GL) theory the Duhamel–Neumann relationships and the entropy relation are modified by introducing two relaxation times that relate the stress and entropy to the temperature rate. An alternative approach in the formulation of a theory predicting the finite propagation speed of the thermal disturbances is due to Green and Naghdi (GN), where they formulated three models of thermoelasticity for homogeneous and isotropic materials [5,6] labeled as models I, II, and III. These theories of thermoelasticity (LS, GL, and GN theories) are known as the generalized theories, or thermoelasticity theories with the second sound effect or with finite thermal wave speed. Ignaczak [7] suggested a combined system of coupled equations for the LS and GL theories. Also, the same author reported a survey of the domain of influence for the results of the LS and GL theories [8]. Francis [9], Ignaczak [7], and Chandrasekharaiah [10,11] have reported brief reviews of these theories.

In situations such as those involving very short transient duration and/or when the sudden high heat flux is applied to a structure, the second sound effect is important. Nayfeh [12] and Nayfeh and Nemat-Nasser [13] used the Lord and

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Shulman theory to study the effects of the thermal coupling on the plane harmonic thermoelastic waves in unbounded media and Rayliegh surface waves propagating along the free surface of a half space. Also, the important role of the coupling term in the response of elastic solids under transient thermal loads is investigated by Amin and Sierakowski [14]. Chen and Dargush [15] used the boundary element method to analyze the transient and dynamic problems in generalized thermoelasticity of a half-space using the Laplace transform method. Chen and Lin [16] employed a hybrid numerical method based on the Laplace transform and control volume method to analyze the transient coupled thermoelastic problems with relaxation times involving a nonlinear radiation boundary condition. Boundary element method is employed by Hosseini Tehrani and Eslami [17,18] for the analysis of coupled thermoelastic problems in a finite domain, where they studied the coupling coefficient effects on the thermal and elastic waves propagation. A transfinite element method is considered by Bagri and Eslami [19] to study the generalized coupled thermoelasticity of an annular disk based on the LS theory. They investigated the thermal and stress waves propagation through the radius of the disk and showed that for thermal shock problems the coupling coefficient has significant effect on the variation of thermal stresses, displacement, and temperature.

Several investigators employed the GN models to solve a variety of thermoelastic problems. The uniqueness of solution of the governing equations for the GN theory formulated in terms of stress and energy-flux is established in Ref. [20]. Chandrasekharaiah [21] studied the one-dimensional thermal wave propagation in a half-space based on the GN model due to a sudden exposure of temperature to the boundary, using the Laplace transform method. Chandrasekharaiah [22] also presented the complete solutions of the governing field equations for the GN theory. Sharma and Chauhan [23] investigated the disturbances produced in a half-space under the application of a point load and thermal source acting on the boundary of the half-space. The material is assumed to be homogeneous and isotropic. The Laplace and Hankel transforms are used and different theories of generalized thermoelasticity are employed to provide a basis to compare the results. Taheri et al. [24] studied the problem of coupled thermoelasticity of a layer based on the GN theory. The problem was transformed into the Laplace domain, where the analytical solution was obtained. An inverse numerical method was then employed to obtain the solution in real time domain.

In this paper, a new unified formulation for the generalized coupled thermoelasticity theories based on the LS, GL, and GN models is proposed. The unifier parameters are introducing to consolidate the equations of the LS, GL, and GN theories into a single system of equations for the anisotropic and heterogeneous materials. The equations are also simplified for the isotropic and homogeneous materials. The reduced equations for the isotropic and homogeneous material are considered and are transformed into the Laplace domain. Analytical solution of the equations are obtained for hollow sphere and cylinder, where a term is introduced to consolidate the solutions for the sphere and cylinder in a unified form. A thermal shock load is applied to the inner surface of the sphere and cylinder, where the solution in analytical form is obtained in the space domain. The numerical inversion of the Laplace transform is then employed to obtain the solution in the time domain.

## 2. Unified formulations of the LS, GL, and GN theories

The fundamental equations of the LS, GL, and GN theories in a unified form are presented here introducing the terms $\eta$ and $t_3$ as unifier parameters. These equations in general form are:

Equations of motion:

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \ddot{\mathbf{u}}, \tag{1}$$

Linear strain–displacement relations:

$$\mathbf{E} = \tfrac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})'), \tag{2}$$

Hooke's law for the linear thermoelastic materials:

$$\boldsymbol{\sigma} = \mathbf{C}\,\mathbf{E} - \boldsymbol{\beta}(T - T_0 + t_1 \dot{T}), \tag{3}$$

Energy balance equation:

$$\nabla \cdot \mathbf{q} = R - T_0 \dot{S}, \tag{4}$$

Entropy relationship:

$$S = \left(\frac{\rho c}{T_0}\right)(T + t_2 \dot{T} - T_0) + \boldsymbol{\beta} : \mathbf{E} - \frac{1}{T_0}\widehat{\mathbf{C}} \cdot \nabla T, \tag{5}$$

Heat conduction equation:

$$\eta\,\mathbf{q} + \eta\,\boldsymbol{\tau}\,\dot{\mathbf{q}} + t_3\,\dot{\mathbf{q}} = -\eta\,\mathbf{K}\,\nabla T - t_3\,\mathbf{K}\,\nabla \dot{T} - t_3\,\mathbf{K}^*\,\nabla T - \widehat{\mathbf{C}}\,\dot{T}, \tag{6}$$

where $\rho$ is the mass density, $\boldsymbol{\sigma}$ is the Cauchy's stress tensor, $\mathbf{u}$ is the displacement vector, $\mathbf{b}$ is the body force vector per unit mass, $\mathbf{q}$ is the heat flux vector, $T_0$ is the reference temperature, $T$ is the absolute temperature, $S$ is entropy per unit volume, $R$ is the internal heat source per unit volume per unit time, $\mathbf{E}$ is the strain tensor, $\boldsymbol{\beta}$ is the second order tensor of stress–temperature moduli, $\mathbf{K}$ is the second order tensor of thermal conductivity, $\mathbf{C}$ is the forth order tensor of elastic moduli, and $c$ is the specific heat, respectively. Also, $\boldsymbol{\tau}$ is the second order tensor of relaxation times in the LS model, $t_1$ and $t_2$ are the relaxation times and $\widehat{\mathbf{C}}$ is a vector of new material constants proposed by Green and Lindsay, and $\mathbf{K}^*$ is the second order tensor of new material constants associated with the GN theory. Also, $\eta$ and $t_3$ are terms introduced to consolidate all theories into a unified system of equations. In Eqs. (1)–(6) the superscript dot (.) denotes differentiation with respect to time. Meanwhile, $\nabla$ is the del operator and indicates the gradient of a function, $(\nabla.)$ denotes the divergence operator, and the superscript prime $(')$ indicates the transpose of the matrix. In Eqs. (1)–(6) the double dot