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# Vibration analysis of non-uniform orthotropic Kirchhoff plates resting on elastic foundation based on nonlocal elasticity theory

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ABSTRACT

The free vibration analysis of non-local orthotropic Kirchhoff plates has been investigated. Kirchhoff plates at the micro/nanoscale are modeled using Eringen's nonlocal elasticity theory, where the small scale effect is taken into consideration. The governing equations are derived using the nonlocal differential constitutive relations of Eringen. For this purpose, the resulted eigenvalue problem is solved numerically by applying the Chebyshev collocation method. The effects of the the Winkler modulus parameter, the shear modulus parameter, the aspect ratio, the taper, the nonlocal scale coefficient, and the boundary conditions on the natural frequencies have been studied.



## 1. Introduction

Micro- and nano-structures have gained appreciated consideration due to their significant role in different engineering and modern technology fields such as aerospace, communications, composites, electronics, microelectromechanical and nanoelectromechanical systems. These structures have superior mechanical, electrical, and thermal properties when compared with other structures at the normal length scale. These properties make them ideal to be used in highly sensitive and high frequency devices for different applications [1]. However, some manufactured micro and nano-structures have non-uniformities in the geometry, such as the variation in the thickness that may affect the dynamical behavior of these structures.

To provide weight reduction for greater structural efficiency, and for more efficient vibration control, tapered beams and plates can be used. Moreover, to increase the strength to weight ratio, orthotropic plates with different fiber orientations are commonly used in several industrial applications such as civil and aerospace structures. Furthermore, these structures may be resting on or embedded in elastic foundations (medium). However, to date, no report has been found in the literature on the vibration response of non-uniform orthotropic nonlocal plates resting on linear elastic foundation. Motivated by these considerations and in order to improve the design of MEMS/NEMS, the aim of this article is to study the free vibration of nonlocal, non-uniform plates resting on elastic foundation.

In the present article, Eringen's nonlocal elasticity theory is applied to study the small-scale effect on the free vibrations of non-uniform orthotropic Kirchhoff plates resting on a Pasternak foundation. The governing equation is derived using Eringen's non-local constitutive relations along with Hamilton's principle. The Chebyshev collocation method is applied for the numerical solution of the governing differential equation, by transforming it to a system of algebraic equations. The system is expressed in a compact form by the Chebyshev differentiation matrices and the use of the Kronecker product. The present formulation allows considering the variable coefficients of the differential equation in a convenient way. In this study, the variable coefficients represent the distribution of the plate's thickness and it's derivatives. The results from the present article can be used for design and optimization of different non-uniform nano-devices embedded in elastic medium. Furthermore, the present analyses may help in the study of vibration response of non-uniform nanodevices (that can be modeled as plates) when used as mechanical resonators and vibrating components.

It is known that the experimental and atomistic simulations and models are capable to show the effects of the small-scale on the mechanical properties of the micro/nanostructures, however, these methods are expensive and restricted by computational

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capacity. Since the local continuum theories for beams (Euler and Timoshenko) and plates (Kirchhoff and Mindlin) are scale free, therefore they are not able to capture the small scale effect on the mechanical, electrical, and thermal properties for very small beam and plate like structures. This makes them inadequate in describing the dynamical behavior for these structures [2]. In order to apply the continuum mechanics approach in the analysis of the micro and nanostructures, logical and reasonable modifications that take into consideration the scale effect, should be proposed. For this purpose, several theoretical models have been suggested. Among these, the strain gradient theory, the modified coupled stress theory, and the nonlocal elasticity theory [3] that will be utilized in this article to analyze the free vibration problem of nonlocal orthotropic Kirchhoff plates resting on elastic foundation.

Many researchers applied the nonlocal elasticity theory to study the free vibration, buckling, deflection, and dynamic problems of micro and nanostructures. For example, Shahidi et al. [4] carried out the transverse vibration analysis of non-uniform orthotropic Kirchhoff plates using the nonlocal elasticity theory and the variational principle. It was concluded that the thickness variation has a noticeable effect on the natural frequencies of the plates. Furthermore, Murmu and Pradhan [5] studied the vibration response of nonuniform cantilever beam utilizing the nonlocal elasticity theory. They applied the differential quadrature method to obtain the natural frequencies. Reddy [6] obtained analytical solutions for the bending, buckling, and vibration problems for simply supported Euler, Timoshenko, Reddy, and Levinson beams using Eringen's nonlocal theory. Murmu and Adhikari [7] studied the nonlocal transverse in-phase and out-of-phase vibrations of double nanobeam systems, in which explicit closed form expressions for natural frequencies were derived. Shakouri et al. [8] applied the Galerkin approach to study the free vibration problem of nonlocal Kirchhoff plates with different boundary conditions. It was shown that the nonlocal parameter and Poisson's ratio have significant effects on the vibration. Wang et al. [2] applied the Hamilton's principle, Eringen's nonlocal elasticity theory, and Timoshenko beam theory to analyze the free vibration problem of micro/nanobeams. Their study concluded that the effects of small scale, rotary inertia, and transverse shear deformation are important on the vibration behavior of short and stubby micro/nanobeams. Other researchers applied the nonlocal elasticity theory to study the scale effect on different dynamical systems [9–20].

## 2. Theory

### 2.1. Chebyshev spectral collocation

The Chebyshev or the Gauss-Chebyshev-Lobatto points are the points that represent the projections on the interval $[-1, 1]$ of equally spaced points of a unit circle. These points are numbered from right to left as shown in Fig. 1 [21], and defined by

$$x_j = \cos(j\pi/N), \; j = 0, 1, \ldots, N. \tag{1}$$

The Chebyshev differentiation Matrix $D_N$ of size $(N+1)\times(N+1)$ can be obtained by interpolating a Lagrange polynomial of degree $N$ at each Chebyshev point, differentiating the polynomial, and then finding its derivative at each Chebyshev point. The entries of the $D_N$ matrix are given as

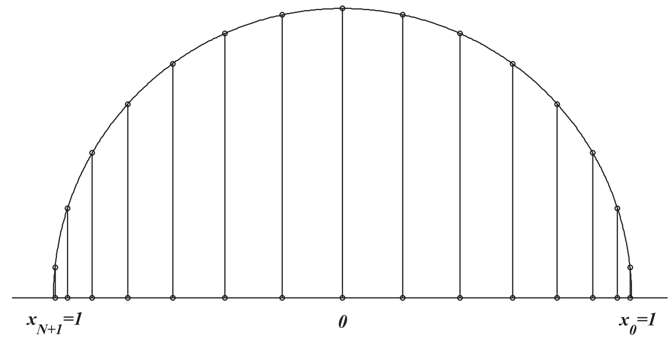


**Fig. 1.** Chebyshev Points.

$$(D_N)_{00} = \frac{2N^2+1}{6}, \; (D_N)_{NN} = -\frac{2N^2+1}{6}, \; (D_N)_{jj} = \frac{-x_j}{2(1-x_j^2)}, \quad j = 1, \ldots, N-1 (D_N)_{ij} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{(x_i - x_j)}, \quad i \neq j, \quad i, j = 0, \ldots, N. \quad c_i = \begin{cases} 2, & i = 0 \text{ or } N, \\ 1, & \text{otherwise}. \end{cases} \tag{2}$$

The Chebyshev collocation method can be applied to solve ordinary or partial differential equations (ODEs or PDEs), by representing the $n^{\text{th}}$ derivative of a function by $Dn=(D_N)^n$.

The Chebyshev collocation method was successfully employed to carry out the free vibration analysis of local continuous systems with different shapes, geometries, and boundary conditions [22–25].

### 2.2. Nonlocal theory

The nonlocal elasticity theory was introduced by Eringen [3] accounts for the small-scale effects arising at the nanoscale level. In his work, the stress at a point is assumed to be as a function of the strains at all points in the domain. Many researchers applied the nonlocal elasticity theory to study the free vibration, buckling, deflection, and dynamic problems of micro and nanostructures. The small-scale effects and the atomic forces become of great importance in designing, optimizing, and improving the performance of micro and nano structure materials. For such materials, the internal length scales of the material are comparable to the structure size. Moreover, a long range cohesive force exists among the particles in addition to the contact forces and the heat diffusion. Consequently, the internal length scale should be considered as a material parameter which is called nonlocal parameter, this parameter should be taken into account in both of the governing equations as well as the constitutive relations.

For non-local linear elastic solids, the stress tensor $t_{ij}$ is defined by

$$t_{ij} = \int_V \alpha(|x' - x|)\sigma_{ij}(x')dV(x') \tag{3}$$

where $x$ is a reference point in the elastic domain, and $\alpha(|x' - x|)$ is the non-local kernel attenuation function. It introduces the nonlocal effects at the reference point $x$ produced by the local stress $\sigma_{ij}$ at any point $x'$, and $|x' - x|$ is the distance in Euclidean form.

In order to simplify Eq. (1), Eringen introduced a linear differential operator $\varsigma$, defined by $\varsigma = 1 - (e_0 l)^2\nabla^2$, in which $e_0$ is a material constant estimated by experiments or other models and theories. The non-local theory relations could result in approximate solutions to those obtained by atomic theory. The value of $e_0$ was taken to be 0.39 in Eringen's analysis. Moreover, the constant$l$represents the internal characteristic length which is of the