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Comprehensive and easy-to-use torsion and bending theories for micropolar beams

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ABSTRACT

The main goal of this paper is to develop a comprehensive beam model based on the micropolar elasticity theory which is as general, as easy to use, and as convenient as the classical beam theories. Uncomplicated torsion and bending theories for micropolar elastic beams deforming in three-dimensional space and under different types of external loading and boundary conditions are presented in this paper. Unlike the classical beam models, the developed beam model includes the effect of microinertia and contains new material parameters to capture the microstructure-dependent size effects which could be useful when dealing with micro scale beams. The presented micropolar beam model generalizes the Duleau torsion and Timoshenko bending beam models to include the microstructure effects. Hamilton's principle and a variational approach are used to derive the dynamic equations of the micropolar beam with longitudinal, torsional, and bending deformations. Then the governing dynamic equations are solved numerically by using a finite element approach and numerical results for a simply supported micropolar beam are provided. The static and dynamic behaviors of the developed micropolar beam model are studied and compared against the classical beam models. In particular, the conditions for recovery of the results of the classical beam theories, *i.e.* Duleau and Timoshenko theories, are addressed.

1. Introduction

Beams are ubiquitous, they occur in a wide range of engineering applications from NEMS and MEMS scale biosensors and atomic force microscopes [1,2] to very large buildings, bridges, and space structures [3–5]. For large scale structural systems classical elasticity theory provides us with good engineering theories for beam bending, *e.g.* Euler–Bernoulli or Timoshenko bending theory [6], and torsion, *e.g.* Duleau torsion theory [7]. For small scale beams the classical beam theories lack the ability to capture the "size effects" [8,9] and they also fail to adequately take into account the asymmetric stress tensor that arises in the presence of a body (volume) moment [10–12]. The microstructure-dependent size effects are negligible at the macro scale but have been documented to be important at the nano and micro scales [13,14]. These observations have resulted in the development of beam models based on continuum theories that can account for the size effects through the use of additional material parameters. Examples include the couple-stress [15,16] and micropolar [17–19] elasticity theories.

Microstructure-dependent beam models based on the couple-stress theory and Eringen's non-local elasticity theory [20] have attracted attention in conjunction with the NEMS and MEMS applications (*e.g.* see [21,22] and their references). These models, however, suffer from a degree of complexity that makes them unattractive for application in engineering problems. Even in NEMS and MEMS systems classical beam theories are still widely relied upon [23,24]. The relative lack of use of micropolar beam theories in these engineering applications is also due to the absence of a versatile formulation which has a broad set of solutions for a range of boundary conditions and loading cases. In other words, existing micropolar beam models for both torsion and bending are usually dedicated to pure bending or torsion of beams with uniform circular or square cross-section subject to a bending or torsion moment at the ends and do not seem as convenient as classical beam torsion and bending models to a practitioner. Treatments of pure bending of micropolar elastic beams [25–27] are typically very complicated and difficult to understand as are treatments of pure torsion [8,28–30]. Not to mention that setting up a finite element approach on these treatment is not a straightforward task. Indeed, despite its importance little research has been found on a finite element formulation for micropolar beams. There is only a preliminary work by Huang et al. [31] that briefly mentions the finite element formulation for an over-simplified micropolar beam bending model.

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The simple micropolar beam bending model developed by Huang et al. [31] is based on the assumptions that shear deformations are negligible and the microrotation over the beam cross-section is constant and equal to the rotation of the beam plane section due to the bending. This model can be considered as an extension of the classical Euler–Bernoulli bending formulation. In a similar approach, by assuming that beam microrotations are constant over the beam cross-section and equal to the classical plane section bending and torsional rotations, Goda et al. [32] developed reduced micropolar torsion and bending theories. However, it should be noted that assuming equal macro- and micro-rotations (*i.e.* equal classical and micropolar rotations) in these models results in reducing the micropolar elasticity theory to the couple-stress theory. Therefore, the models in [31,32] may not be considered as micropolar beam models and are rather couple-stress beam theories.

Ramezani et al. [33] have presented a more general but still simplified micropolar beam bending model that is a generalization of the Timoshenko bending model. They assumed that the microrotation is constant over the beam cross-section but is different from the bending rotation of the beam plane section. Unfortunately, both Haung et al. and Ramezani et al. used an incorrect definition for the strain tensor in terms of displacement and microrotation fields early in their derivation of the governing equations and therefore their results are questionable. The erroneous strain definition in these two papers can be appreciated by comparing the first equation of each paper with the strain definition in any of [8,28,29]. Finally, Aganović et al. [34] used an asymptotic approach to derive a simple beam model from micropolar elasticity theory. The final set of equations are however questionable (particularly for torsional deformations), no boundary conditions are derived, and no numerical results are presented.

The purpose of this paper is to present the development of a simplified but otherwise comprehensive beam model that can simultaneously deal with general axial deformation, bending, and torsion while using a micropolar elasticity material formulation that accounts for microinertia effects, and microstructure-dependent size effects. The bending behavior is captured by a generalized form of Timoshenko bending theory and the torsional behavior is characterized by an extended form of Duleau torsion theory. The developed micropolar beam model allows for an effortless direct set up of a finite element formulation in the weak form to solve the beam equations in both static and dynamic cases. To illustrate the developed beam model, the static and dynamic behaviors of a simply supported micropolar beam are studied.

The paper is organized as follows. After presenting the notation in Section 2, the fundamental equations of the linear micropolar elasticity are reviewed in Section 3 and the kinematic assumptions for derivation of the micropolar beam model are addressed in Section 4. Then the potential and kinetic energy expressions and the virtual work expression are obtained in Sections 5–7. Hamilton's principle is then applied to these expressions to derive the governing dynamic equations and boundary conditions given in Section 8. Numerical solution of the derived equations by employing a finite element approach is addressed in Section 9. Example results are reported and compared against those of the classical beam models in Section 10. Finally, the summary and conclusions are given in Section 11.

2. Notational conventions

In this paper Cartesian tensor concepts and accompanying index notations are used where it is meaningful. Generally, small Latin subscripts i, j, and k take the values 1–3 (unless mentioned otherwise). For expressions with repeated Latin subscripts, the Einstein summation convention over that subscript, from 1 to 3 , is understood. A comma followed by a subscript denotes partial differentiation with respect to the corresponding Cartesian coordinate (see [35,36]), *i.e.* $z_{,i}$ is equivalent to $\frac{dz}{dx_i}$.

The elements of a vector (first-order tensor) v and a dyadic (second-order tensor) d, described in a frame $\mathcal{F}_A$, are shown as ${}^a v_i$ and ${}^a d_{ij}$ respectively. In general, a leading superscript denotes the name of the frame in which a description or operation is done. This leading superscript is omitted when the inertial frame $\mathcal{F}_o$ serves as the reference frame. A matrix is denoted as $\underset{\sim}{\boldsymbol{m}}$, its transpose is denoted as $\underset{\sim}{\boldsymbol{m}}^{\mathrm{T}}$, and its elements are referred to by m_{ij}.

3. A review of linear micropolar elasticity

Although well developed and verified [37] the classical theory of linear elasticity fails to produce acceptable results for the cases with large stress gradients or for materials with significant microstructure contribution (*e.g.* composites, polymers, soil, and bone, especially when the dimensions involved in the problem are relatively small) [19]. It is not an appropriate theory for asymmetric stress–strain analysis (as may arise with an elastic body under the action of a volume moment distribution *e.g.* gyroelastic materials [12,38] and magnetized materials [10]). For these cases newer material models such as couple-stress theory and micropolar elasticity theory are more suitable.

The origins of micropolar elasticity begin with Voigt's work [39] on adding an independent couple stress vector to the classical force stress vector to describe the interactions between neighboring elements of an elastic body. Voigt's theory, known as couple-stress elasticity, was further developed by E. and F. Cosserat [40] who suggested independent displacement and microrotation field vectors. Their assumptions lead to six degrees of freedom (DOFs) for every element of the body and a description of strain and stress in terms of asymmetric tensors. The Cosserat theory of elasticity was further developed by Eringen [18] who extended the theory to include microinertia effects.

In micropolar elasticity the conventional displacement field u_i and the force stress tensor σ_{ij} are complemented by an independent microrotation field ϑ_i and a couple stress tensor χ_{ij}. The body deformation is characterized in terms of the potentially asymmetric strain tensor ϵ_{ij} and twist tensor τ_{ij} which are defined as:

$$\epsilon_{ij} = u_{j,i} - \varepsilon_{ijk}\vartheta_k, \quad \tau_{ij} = \vartheta_{j,i}, \tag{1}$$

where ε_{ijk} is the third-order Levi-Civita or permutation tensor.

The micropolar elasticity constitutive relations that relate σ_{ij} and χ_{ij} to τ_{ij} and τ_{ij} are presented in terms of two relations containing six elastic constants, specifically:

$$\begin{aligned} \sigma_{ij} &= \left(\mu + \kappa\right)\epsilon_{ij} + \left(\mu - \kappa\right)\epsilon_{ji} + \lambda\epsilon_{kk}1_{ij}, \\ \chi_{ij} &= \left(\gamma + \beta\right)\tau_i j + \left(\gamma - \beta\right)\tau_{ij} + \alpha\tau_{kk}1_{ij}, \end{aligned} \tag{2}$$

where 1_{ij} is the Kronecker delta tensor. The six elastic constants are the Lamé coefficients μ and λ (μ is the shear modulus), and four extra micropolar elastic constants κ, γ, β, and α representing the contribution of the material microstructure to the elastic properties of the body. The micropolar couple modulus κ couples the displacement and microrotation DOFs to each other. The ratios $(\beta/\mu)^{1/2}$ and $(\gamma/\mu)^{1/2}$ represent the two length scales of the micropolar material, and the ratio $\alpha/2(\gamma + \alpha)$ is the micropolar Poisson's ratio relating the transverse curvature to the principal curvature. Finally, based on the linear theory of micropolar elasticity, the total strain energy $\mathcal{U}$ and the kinetic energy $\mathcal{U}$ take the form: