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Unified formulation of the stress field of saint-Venant's flexure problem for symmetric cross-sections

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ABSTRACT

The flexure problem of Saint-Venant's elastic beam under lateral traction is revisited. First an engineering stress field is proposed with the explicit closed form solution that results in shear stress distribution determined by mechanics of materials theory. Afterward a modified stress field is introduced for Saint-Venant's flexure problem with uniaxial symmetric cross-sections that recovers the solutions available from the theory of elasticity. The modified stress field which is a unified formulation of Saint-Venant's solution confirms the main features of shear stress distribution found in the earlier investigations and has an excellent agreement with the results of the theory of elasticity. Also the shear flexibility factor is comprehensively discussed and an explicit solution form is presented based on the modified stress field.

1. Introduction

Saint-Venant was the first one who studied the problem of linear elastostatics for a right prismatic beam under the action of a single transverse load [1]. Although Saint-Venant's solution method stands as one of the supreme monuments of applied mechanics but the exact solution for determination of elastic shear stress field in arbitrary shape cross-sections is deficient [2]. The original solution due to Saint-Venant is to obtain the plane harmonic functions satisfying a certain condition round the boundary of elliptic, circular and rectangular cross-sections. Afterward solutions for other shapes of cross-sections have been obtained utilizing polar and curvilinear coordinates [3,4] as well as the application of conformal mappings [5,6]. However solution of Saint-Venant's problem for a triangle of arbitrary geometry, in terms of shear stresses, is not known yet. For particular value of Poisson's ratio, Timoshenko [7] gives the flexure solution of isosceles triangle cross-sections; later on Seth [8] dealt with cross-section of right-angled isosceles triangle and triangular cross-sections with arbitrary direction of shear force [9]. Seth [10] also obtained a simple solution for Saint-Venant's problem when the boundary is a rhombus but the solution was merely valid for specific geometry as a function of Poisson's ratio. The Saint-Venant's problem is still on major focus of researchers investigating new material behaviors including non-homogeneous and anisotropic elasticity [11,12], linear piezo-elasticity [13,14] and functionally graded materials [15,16]. Moreover the Saint-Venant's problem with general loading conditions is studied by Barber [17] developing a symmetric solution procedure. Lacarbonara and Paolone [18] also comparatively discussed different solution strategies to the relaxed Saint-Venant's problem. In a series of recent publications by Barretta [2,19–22], various aspects of the Saint-Venant's problem have been also investigated by a coordinate-free approach also for composite orthotropic beams [23].

In present study the flexure problem of Saint-Venant's prismatic beam under loading of lateral shear force is revisited. A semi-inverse method is used to introduce a closed form solution for engineering stress field. The solution confirms the distribution of shear stresses for symmetric cross-sections in the sense of the mechanics of materials theory. Furthermore based on the proposed engineering stress field, a modified stress field is introduced for cross-sections possessing uniaxial symmetry. The introduced modified stress field is shown to be a unified formulation of Saint-Venant's problem that confirms the solutions found in the earlier investigations based on the theory of elasticity. The assessment of modified stress field would be done by recovering the solutions available from the theory of elasticity for a set of six uniaxial symmetric cross-sections. Furthermore, a complete discussion on the shear flexibility would be performed and a closed form solution for the shear flexibility would be presented based on the results of the modified stress field.

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2. Engineering and modified stress field

The flexure problem of prismatic beams with general cross-section made of linear elastic, isotropic and homogeneous material is reconsidered here. It is assumed that the region bounded by the boundaries of the cross-section is simply connected and the origin of Cartesian coordinates coincides with the cross-sectional centroid. The prismatic beam is then subjected to a lateral force V at the end, parallel to x-axis. The geometry of the beam and the applied lateral force and boundary conditions are shown in Fig. 1.

Following the semi-inverse method of Saint-Venant, some certain assumptions should be made regarding the stress field. It is assumed that in-plane stress components $\{\sigma_x, \sigma_y, \tau_{xy}\}$ are vanishing terms and the normal stresses over the cross-section at distance z are distributed as nonsymmetrical bending theory [24],

$$\sigma_z = -\frac{V(L-z)}{I_{xx}I_{yy}-I_{xy}^2}\left(xI_{xx}+yI_{xy}\right) \tag{1}$$

where I_{xx}, I_{yy} and I_{xy} are the moments and product of inertia of the cross-section, respectively.The shear stress acting on the cross-section is assumed to be resolved at each point into components of τ_{xz} and τ_{yz}. With these assumptions, neglecting body forces, the stress equilibrium equations are described as,

$$\begin{aligned}&\frac{\partial\tau_{xz}}{\partial z}=0\\&\frac{\partial\tau_{yz}}{\partial z}=0\\&\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}=-\frac{V\left(xI_{xx}+yI_{xy}\right)}{I_{xx}I_{yy}-I_{xy}^2}\end{aligned} \tag{2}$$

It is clearly inferred from the first two equations of stress equilibrium that the shear stress components of τ_{xz}, τ_{yz} are independent of z and consequently would be the same in all cross-sections of the beam. To balance the third equation of stress equilibrium, an appropriate particular solution for the shear component τ_{xz} could be a quadratic and linear function of x and y, respectively. Thus the shear component τ_{xz} would be expressed as,

$$\tau_{xz}=\frac{V}{k\left(I_{xx}I_{yy}-I_{xy}^2\right)}\left[(c_1-x)(c_2+x)I_{xx}+yI_{xy}\right] \tag{3}$$

where c_1, c_2 are the extreme heights of the cross-section and k is the correction factor. The correction factor would be determined by satisfying the force balance across the domain in the x direction as,

$$k=\frac{I_{xx}\left(c_1c_2A-I_{yy}\right)}{I_{xx}I_{yy}-I_{xy}^2} \tag{4}$$

where A is the area of the cross-section. Imposing the proposed form of shear stress τ_{xz} into the third equation of stress equilibrium equations, the particular solution for shear component of τ_{yz} may be shown to be,

$$\tau_{yz}=\frac{Vy}{2k\left(I_{xy}^2-I_{xx}I_{yy}\right)}\left[\left((k-2)x+c_1-c_2\right)2I_{xx}+kyI_{xy}\right] \tag{5}$$

However since the product of inertia vanishes for cross-sections with uniaxial symmetry, the correction factor would be reduced to,

$$k=\frac{c_1c_2A}{I_{yy}}-1 \tag{6}$$

Moreover out-of-plane stress components for uniaxial symmetric cross-sections would be simplified as,

$$\begin{aligned}\sigma_z&=-\frac{Vx(L-z)}{I_{yy}}\\\tau_{xz}&=\frac{V}{kI_{yy}}(c_1-x)(c_2+x)\\\tau_{yz}&=-\frac{Vy}{kI_{yy}}\left[(k-2)x+c_1-c_2\right]\end{aligned} \tag{7}$$

The proposed distribution of shear stress τ_{xz} is uniform in y direction that is similar to the results of the mechanics of materials theory. Although the out-of-plane stresses do not match the exact elasticity solution but will result in the engineering stress distributions which have a good estimation of its true values [25]. In Section 4, the level of accuracy of the proposed engineering stress field would be examined and compared to the classic approach of mechanics of materials theory for a set of six convex symmetric cross-sections.

Alternatively to obtain the elasticity solution, the stress components must satisfy the Beltrami–Mitchell compatibility equations in the absence of body forces. Thus motivated from mechanics of materials theory, the normal stress component of σ_z is assumed to be distributed as symmetrical flexure theory and the system of compatibility equations would be reduced to [24],

$$\begin{aligned}\nabla^2\tau_{yz}&=\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{yz}}{\partial y^2}=0\\\nabla^2\tau_{xz}&=\frac{\partial^2\tau_{xz}}{\partial x^2}+\frac{\partial^2\tau_{xz}}{\partial y^2}=-\frac{V}{I_{yy}(1+\nu)}\end{aligned} \tag{8}$$

where ν denotes the Poisson's ratio of linear elastic materials. Also the lateral surface of the prismatic beam is assumed to be free from external forces. Consequently the traction free boundary conditions are reduced to

$$\tau_{xz}\frac{dy}{ds}-\tau_{yz}\frac{dx}{ds}=0 \tag{9}$$

where ds is an element of the bounding curve of the cross-section. It is noteworthy that the traction free boundary conditions would result in a tangential resultant shear stress to boundary of the cross-section. Therefore the elasticity solution of the flexure problem of a prismatic beam with uniaxial symmetric cross-section would be reduced to solve stress equilibrium equations, Eq. (2), compatibility equations, Eq. (8), and traction free boundary conditions, Eq. (9). Hence based on the proposed engineering stress distribution, the modified stress distribution would be proposed as,

$$\begin{aligned}\sigma_z&=-\frac{Vx(L-z)}{I_{yy}}\\\tau_{xz}&=\frac{V}{kI_{yy}}\left[(c_1-x)(c_2+x)+\Lambda(y)\right]\\\tau_{yz}&=-\frac{Vy}{kI_{yy}}\left((k-2)x+c_1-c_2\right)\end{aligned} \tag{10}$$

where the function of $\Lambda(y)$ governs the behavior of the shear stress component of τ_{xz} in y direction. The modified stress field would obviously satisfy the stress equilibrium equations and it may be shown that a particular solution for the function of $\Lambda(y)$ should have the following quadratic form in terms of y to satisfy the compatibility equations,

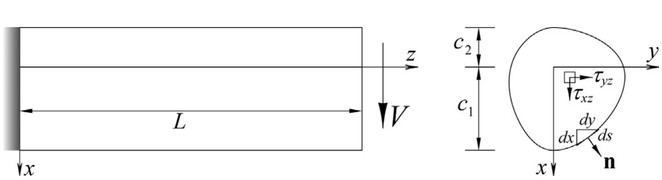

Fig. 1. The geometry and loading conditions of prismatic beam.