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The application of plastic flow theory to inelastic column buckling

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ABSTRACT

The paper presents a theory of inelastic column buckling which is consistent with the principles of plastic flow theory. The theory accounts for flexural, torsional and flexural-torsional modes. While the use of the tangent modulus to describe inelastic flexural buckling has been common place for a long time, efforts to comprehensively unite the torsional and flexural-torsional modes with the principles of plastic flow theory have so far been hampered by the 'plastic buckling paradox'. New theoretical developments presented in this paper provide a way to achieve this goal. The solution hinges on the derivation of the inelastic shear stiffness while considering an infinitesimal solid element embedded within the column at a stage immediately past the point of buckling.

The proposed inelastic column theory is verified against selected experimental data pertaining to aluminium and stainless steel columns of various cross-sections. Particular attention is paid to the torsional buckling problem of the inelastic cruciform section column.

1. Background

With respect to inelastic flexural buckling of columns, Engesser [9] was the first to propose the use of the tangent modulus E_t to predict the buckling load of an initially perfectly straight, inelastic column by modifying Euler's differential equation as follows:

$$E_t I \frac{d^2 u}{dx^2} + Pu = 0 \tag{1}$$

where u is the lateral column deflection, P is the axial compressive load and I is the second moment of area of the cross-section about the principal axis about which bending takes place. Eq. (1) results in an expression for the column buckling load:

$$P_{cr} = \pi^2 \frac{E_t I}{L_e^2} \tag{2}$$

where L_e is the effective length, dependent on the boundary conditions.

While straightforward, Engesser's approach received criticism from Considère [7] who argued that, as the column starts to bend out laterally, elastic unloading takes place on the convex side of bending and that consequently, the bending stiffness is not simply determined by $E_t I$. Engesser [10] replied by proposing his "double-modulus" or "reduced modulus" theory, where:

$$P_{cr} = \pi^2 \frac{E_r I}{L_e^2} \tag{3}$$

with:

$$E_r = E_t \frac{I_c}{I} + E_0 \frac{I_t}{I} \tag{4}$$

I_c and I_t are the second moments of area of the parts of the cross-section subjected to compression and tension with respect to the neutral axis, respectively, and E_0 is the initial elastic modulus.

It soon became apparent that Eq. (2) showed much better agreement with the experiment than Eq. (3), which consistently led to overestimates. Shanley [32] shed light on this seeming paradox by pointing out that Eq. (2) does indeed constitute the buckling load of the column since it indicates the point of bifurcation above which the column cannot be in a state of stable equilibrium while remaining straight. Moreover, lateral buckling does not take place under a constant load, but elastic unloading on the convex side instead results in postbuckling capacity.

A realistic theory describing buckling of inelastic columns involving torsion, which may either occur as pure torsional buckling or combined flexural and torsional buckling, based on the principles of plastic flow theory has not yet been presented. The challenge thereby lies in modelling the relationship between increments of shear stress and shear strain at the onset of buckling. A previous interpretation of plastic flow theory [16,2,23] has suggested that the increments of shear stress and shear strain remain linked through the elastic modulus E_0 at the onset of buckling and

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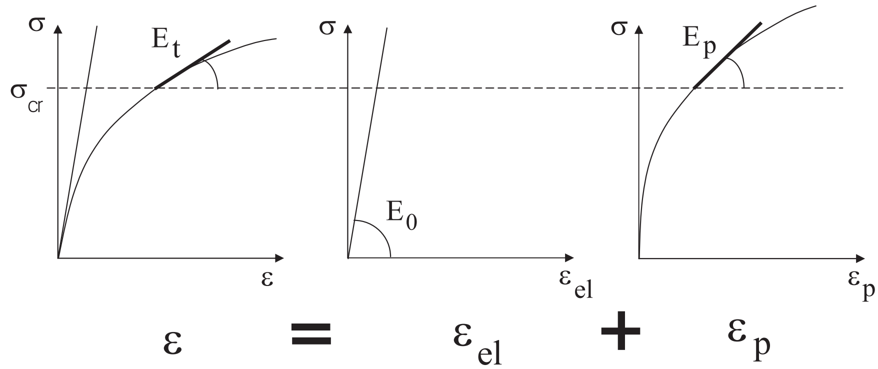

Fig. 1. Material stress–strain curve.

that, therefore, the torsional resistance remains unaffected by the axially induced plasticity. This conclusion, however, stands in clear contradiction with experimental observations, as demonstrated by, among others, [1], Onat and Drucker [25], [2,16,23]. This "plastic buckling paradox", as it is often named in literature, is particularly exemplified by the torsional buckling problem of the inelastic cruciform column, since this particular cross-section relies on the shear stresses resulting from pure torsion to a much larger extent than on the (negligible) longitudinal warping stresses in its buckling resistance. Experiments on cruciform sections have indicated that plastic flow theory substantially overestimates the buckling load. On the other hand, buckling theories based on plastic deformation theory, which is generally considered flawed and inferior in its concept to plastic flow theory, have so far yielded the better predictions in relation to column buckling problems involving torsion. This is more generally true for inelastic bifurcation problems and this paradoxical issue continues to hamper theoretical stability research, as demonstrated recently by, for instance, Rønning et al. [29] for plates, Shamass et al. [31] for cylindrical shells and Ruocco [30] for instabilities in thin-walled elements in general. The plastic buckling paradox has also been excellently illustrated for thick and thin plates under uniaxial, biaxial and shear loading by Wang and Tun Myint [38], Wang et al. [39] and Wang and Huang [40]. The problem also arises within the context of Generalized Beam Theory (GBT), as demonstrated by Gonçalves and Camotim [11]. The authors developed two GBT formulations, incorporating either deformation theory or flow theory. These new formulations were then applied to the cases of simple plates under uniform compression and hat section beams in uniform bending. It was concluded that the flow-based GBT resulted in much higher predictions of the buckling stresses than the deformation-based theory.

Interestingly, it has been observed Shamass et al. [31] that the results of geometrically non-linear finite element analyses using flow theory with an associated flow rule are unaffected by the plastic buckling paradox. While no explanation has yet been provided as to why an incremental numerical approach remedies the problem, a firm conclusion can be drawn from this observation, namely that the plastic buckling paradox is not due to any inherent shortcomings or limitations of flow theory itself, but rather a result of an incorrect application of its principles. This idea is central to the theory proposed in this paper.

Onat and Drucker [25] demonstrated that the plastic buckling paradox can be circumvented by incorporating imperfections into the model and that even very small, inevitable imperfections have a severe impact on the buckling load, reducing it to levels close to those predicted by deformation theory. Hutchinson and Budiansky [17] confirmed this finding for low strain-hardening metals. However, they also demonstrated that for metals with significant strain-hardening the imperfections have to be of such magnitude that they can no longer be considered 'small and inevitable', thus suggesting that Onat and Drucker's explanation is not entirely satisfying.

The approach presented in this paper differs from the aforementioned rationale in that a perfectly straight column is considered, without initial imperfections. Instead, the plastic buckling paradox is resolved by deriving a relationship between shear stress and shear strain increments at the onset of buckling, while applying the plastic flow rule to a solid element in its shear deformed shape. The basic principles of plastic flow theory, however, are retained.

2. Inelastic shear stiffness

An expression for the inelastic shear stiffness G_1 of a non-linear metal is first derived, accounting for the presence of a uniaxial compressive stress. G_1 will be used in the following paragraphs to relate increments of shear stress and shear strain at the point of column buckling. The derivation here presented is a generalized and amended version of the one contained in Becque [4].

We consider the material stress-strain curve of a non-linear metal, as determined from a uniaxial compression test (Fig. 1). It is a generally accepted postulate of plasticity that an increment in axial strain $\dot{\varepsilon}_1$ is composed of an reversible elastic component $\dot{\varepsilon}_{1,el}$ and an irreversible plastic component $\dot{\varepsilon}_{1,p}$:

$$\dot{\varepsilon}_1 = \dot{\varepsilon}_{1,el} + \dot{\varepsilon}_{1,p} \tag{5}$$

Eq. (5) can be written in terms of the increment in axial stress $\dot{\sigma}_1$associated with $\dot{\varepsilon}_1$:

$$\frac{\dot{\sigma}_1}{E_t} = \frac{\dot{\sigma}_1}{E_0} + \frac{\dot{\sigma}_1}{E_p} \tag{6}$$

where E_0 is the elastic modulus, E_t is the tangent modulus at the relevant stress level and E_p relates the plastic stress and strain increments. Thus:

$$\frac{1}{E_p} = \frac{1}{E_t} - \frac{1}{E_0} \tag{7}$$

Plastic flow theory also dictates that the incremental plastic strain in the perpendicular principal 2-direction is given by:

$$\dot{\varepsilon}_{2,p} = \kappa \dot{\varepsilon}_{1,p} \tag{8}$$

An associated flow rule is adopted, so that κ in Eq. (8) is determined by the slope of the normal to the flow surface [8]. When the von Mises surface is used (Fig. 2), κ amounts to $-1/2$ under uniaxial compression. However, the calculations will carry a general κ value to allow for possible plastic anisotropy in the material.

Fig. 3a depicts an infinitesimal element of material embedded