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A new displacement-based solution for thick isotropic curved tubes

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ABSTRACT

Simple beam theories may be applied to straight structures, while the behavior of curved structures subjected to mechanical loadings is complex. In the present study, a displacement approach of Toroidal Elasticity is chosen to analysis thick isotropic curved tubes under pure bending moments. The governing equations are developed in a toroidal coordinate system. The method of successive approximation is used to find the general solution. The accuracy of the presented method is subsequently verified by comparing the results with finite element method (FEM) and stress-based Toroidal Elasticity (SBTE). The results show good agreement. Also, high efficiency in terms of computational time is shown when the presented method is used as compared with FEM (ANSYS). Finally, several numerical examples of stress distributions in the thick isotropic curved tube subjected to pure bending are presented and discussed.

1. Introduction

Stress analysis of curved structures is often a complex task. In addition, governing equations for curved structures are much more complicated comparing with straight structures. Depending on the geometric parameters, the flexibility of curved structures and stresses can be much greater than those predicted by the beam theory. Curved structures exhibit complex deformation fields given their toroidal geometry and the multiplicity of configurations of external loads. One type of these structures is curved tube. Curved tubes are frequently used by the aerospace, offshore and infrastructure industries. Prediction of the state of stress and strain in a curved tube is of theoretical interest and practical importance.

Von Karman [1] found a theoretical explanation for the phenomenon of a curved tube having more flexibility in bending than a straight one. The particular case of the Karman problem, the so-called Brazier effect, which included the buckling analysis of straight or curved pipes, was more complex [2]. Their works provided the fundamentals for much of the subsequent tube analysis.

Ting [3,4] and Chen et al. [5] studied a cylindrical anisotropic circular tube subjected to pressure, shear, torsion and extensive loads for axisymmetric deformation of a homogeneous tube assuming the stresses are just a function of radial distance. A theoretical analysis for the prediction of the ultimate bending strength for tubes subjected to bending was presented by Mamalis et al. [6]. Boyle [7] used nonlinear theory of shell to formulate the tube bending problem. Reissner [8] represented the finite-bending theory for curved tubes. Axelrad and Emmerling [9] studied the flexure of cylinders and slightly curved pressurized tubes. They used the Flexible-Shell-Theory to determine the large pre-critical deformation. Emmerling [10] determined the nonlinear deformation of elastic curved tubes subjected to bending loads. He, also, studied the pre-critical deformation of the tubes on the basis of the semi-membrane theory. Bushnell [11] studied types of instability, and classical buckling problems involving the axially compressed cylindrical shell and the externally pressurized spherical shell.

The developing theory of Toroidal Elasticity (TE) was employed to determine the stress and displacement fields in toroidal rings, elbows and vessels which are utilized in chemical, nuclear and power supply plant. Gohner [12] apparently was the first to investigate the technical problem of a curved solid circular ring sector subjected to pure twist and bending moments. The problem of helical springs subjected to tension and torsion loading was observed by Ancker and Goodier [13] through the thin-slice method. They assumed that the springs had the same cross section and same resultant force and moments on each cross section. Kornecki [14] and McGill [15] developed the theory of Toroidal Elasticity by extending Gohner's work. Kornecki [14] employed the method of successive approximation to solve the governing equations. The finite difference method was used by McGill [15]. A major contribution to the theory of Toroidal Elasticity was done by Lang [16]. He summarized prior research works [16,17] and developed the theory of Toroidal Elasticity in the non-classical Toroidal coordinate system. In Lang's works, the stress approach method was utilized, thus, he did not find deformation fields directly. The works of Lang had been advanced by Redekop [18,19]. He used the

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Nomenclature

a, b	inside and outside cross-sectional radius of the curved tube
c	reference length
a_n, b_n, c_n, d_n, e_n, f_n	constants of the nth order complementary solution
A_{ni}, B_{ni}, C_{ni}	constants of the nth order particular solution
E	Young's modulus
G	shear modulus
M_0	bending moment
R	mean radius of isotropic curved tube
u_ς, u_ϕ, u_θ	displacement components in toroidal coordinate
u, v, w	non-dimensional displacement components
u_k, v_k, w_k	the kth order non-dimensional displacement components
U, V, W	the first part of Navier function
$\overline{U}$, $\overline{V}$, $\overline{W}$	the second part of Navier function
$\hat{U}$, $\hat{V}$, $\hat{W}$	the third part of Navier function
U_k, V_k, W_k	the first part of Navier function of the kth order
$\overline{U}_k$, $\overline{V}_k$, $\overline{W}_k$	the second part of Navier function of the kth order
$\hat{U}_k$, $\hat{V}_k$, $\hat{W}_k$	the third part of Navier function of the kth order
ε	b/R
ν	Poisson ratio
ς, ϕ, θ	toroidal coordinates
$\tau_{\varsigma\varsigma}, \tau_{\phi\phi}, \tau_{\theta\theta}$	non-dimensional normal stress components
$\tau_{\varsigma\phi}, \tau_{\phi\theta}, \tau_{\varsigma\theta}$	non-dimensional shear stress components
$\tau_{\varsigma\varsigma k}, \tau_{\phi\phi k}, \tau_{\theta\theta k}$	the kth order non-dimensional normal stress components
$\tau_{\varsigma\phi k}, \tau_{\phi\theta k}, \tau_{\varsigma\theta k}$	the kth order non-dimensional shear stress components

displacement components as the basic variables and developed the governing Navier equations in this coordinate system. Eric [20] investigated the linear problem of pure bending of thin walled curved tubes. The solution of a circular cylindrical shell under a uniformly pressure was employed as the first approximation to the solution of this problem. Bushnell [21] analyzed an initially uniformly curved tube. The pipe was treated as part of a toroidal shell. Recently, a hybrid formulation solution was used to obtain the stress intensity factor to double-curvature pipe close to the crack area [22]. A formulation dealing with finite shell elements was presented to solve the problem of stress analysis of curved pipes subjected to in-plane bending forces [23]. Kolesnikov [24] considered a short sector of torus as a curved tube and analyzed the large pure bending deformations of the tube. The solution was based on finite curved elements. Finally, Levyakov [25] studied nonlinear equations of in-plane bending of curved tubes based on Reissner's formulation in terms of two unknown functions and parameters. To solve the equations, a numerical method based on the finite-difference approximations and Newton–Raphson iteration technique was developed. The buckling phenomenon for a straight pipe under a pure bending moment was studied by nonlinear FEA [26].

Although finite element methods can be used for analyzing curved structures, it is necessary to do the meshing for each structure every time some dimensions are changed. Therefore, it is desired to have an analytical method where the input to obtain the solution is simple; i.e. one only needs to enter in the actual dimensions without the meshing work. The present study deals with studying analytically the stresses within isotropic curved tubes under pure bending. The displacement-based Toroidal Elasticity (DBTE) which includes the full three-dimensional constitutive relations, is employed. The comparison is done between results obtained from introduced analytical method with SBTE and FEM (ANSYS). Good agreement is obtained. Finally, the stress distributions in tube cross sections are studied through a number of examples. In addition, effects of a geometric parameter on stress distributions are analyzed.

2. Displacement based toroidal elasticity *(governing equations)*

The non-classical toroidal coordinate system is shown in Fig. 1. The isotropic curved tube has a bend radius R, and an annular cross section bounded by radii a and b (see Fig. 1). A general point P in a constant thickness curved tube can be represented easily by the non-classical Toroidal coordinate system r, φ, and θ where r and φ are polar coordinates in the plane of the tube cross section and θ defines the

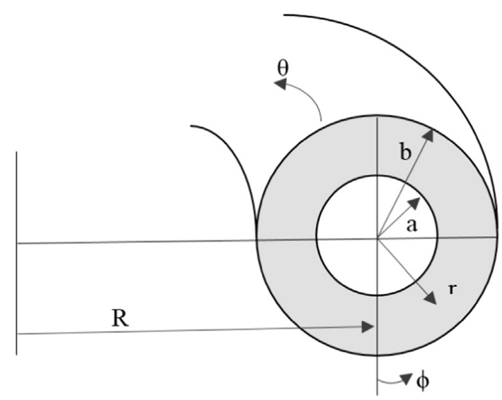

Fig. 1. Geometry and coordinate system of the curved tube.

position of the tube cross section. $\varsigma = r/c$ is a non-dimensional radial coordinate in the toroidal coordinate, and c is a reference length to be specified later. For convenience, non-dimensional displacements u, v, and w are defined as:

$$u = \frac{u_\varsigma}{c}, \quad v = \frac{u_\phi}{c}, \quad w = \frac{u_\theta}{c} \tag{1}$$

Non-dimensional stress components are defined as [18,19]:

$$\tau_{\varsigma\varsigma} = \tfrac{1+\nu}{E}\sigma_{\varsigma\varsigma}; \quad \tau_{\phi\phi} = \tfrac{1+\nu}{E}\sigma_{\phi\phi}; \quad \tau_{\theta\theta} = \tfrac{1+\nu}{E}\sigma_{\theta\theta}$$
$$\tau_{\varsigma\phi} = \frac{1+\nu}{E}\sigma_{\varsigma\phi}; \quad \tau_{\varsigma\theta} = \frac{1+\nu}{E}\sigma_{\varsigma\theta}; \quad \tau_{\phi\theta} = \frac{1+\nu}{E}\sigma_{\phi\theta} \tag{2}$$

The non-dimensional stress components may be expressed in terms of the three non-dimensional displacement components as:

$$\tau_{\varsigma\varsigma} = \tfrac{\partial u}{\partial \varsigma} + \tfrac{\nu}{1-2\nu}\vartheta, \quad \tau_{\varsigma\phi} = \tfrac{1}{2}\left(\tfrac{1}{\varsigma}\tfrac{\partial u}{\partial \phi} + \tfrac{\partial v}{\partial \varsigma} - \tfrac{v}{\varsigma}\right)$$
$$\tau_{\phi\phi} = \left(\frac{u}{\varsigma} + \frac{1}{\varsigma}\frac{\partial v}{\partial \phi}\right) + \frac{\nu}{1-2\nu}\vartheta, \quad \tau_{\varsigma\theta} = \frac{1}{2}\left(\frac{\partial w}{\partial \varsigma} + \frac{c}{\rho}\frac{\partial u}{\partial \theta} - \frac{c}{\rho}w\cos\phi\right)$$
$$\tau_{\theta\theta} = \frac{c}{\rho}\chi + \frac{\nu}{1-2\nu}\vartheta, \quad \tau_{\phi\theta} = \frac{1}{2}\left(\frac{1}{\varsigma}\frac{\partial w}{\partial \phi} + \frac{c}{\rho}\frac{\partial v}{\partial \theta} + \frac{c}{\rho}w\sin\phi\right) \tag{3}$$

where

$$\vartheta = \psi + \tfrac{c}{\rho}\chi$$
$$\psi = \frac{\partial u}{\partial \varsigma} + \frac{u}{\varsigma} + \frac{1}{\varsigma}\frac{\partial v}{\partial \phi}$$
$$\chi = u\cos\phi - v\sin\phi + \frac{\partial w}{\partial \theta} \tag{4}$$