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Nonlinear analysis of nonprismatic Timoshenko beam for different geometric nonlinearity models

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ABSTRACT

This paper deals with the statics and stability of a nonprismatic beam described according to Timoshenko theory, with various geometric nonlinearity models taken into account. The investigated models differed in the complexity of the nonlinear part of the strain tensor. In the simplest model only one nonlinear deformation component, i.e. $(W')^2/2$, was taken into account. Most of the works on geometrically nonlinear beam models, known to the present authors, analyze the simplest model. As demonstrated here this model yields correct results only in for beams with nonslidable supports. An analysis of slidable systems carried out in this paper indicates big differences between the solutions obtained using the different nonlinearity models and shows that in the case of the simplest model the solutions differ considerably from the ones obtained by, e.g., FEM. It also shown that when certain additional strain tensor elements are taken into account, this, although correct from the mathematical point of view, leads to incorrect solutions. One original contribution of this paper is the application of the approximation method to solve the nonlinear problem. The method uses the Chebyshev series whose expansion coefficients are determined from a certain system of recurrence equations. The method enables one to solve equations with variable coefficients. As shown in the previous papers by the author, in the case of solutions to linear problems this method leads to very accurate (also in comparison with analytical solutions) results. The other original contribution is the demonstration of the influence of the particular nonlinear strain tensor components on the solutions to the analyzed problems.

1. Introduction

The analysis of static and dynamic problems in mechanics, with geometrical nonlinearity taken into consideration, has been the subject of numerous investigations and papers. Since it is (usually) impossible to obtain an exact analytical solution to the differential equations describing the problems, the latter are interesting from the theoretical point of view. The analysis of the problems is also important from the practical point of view since there is a need to more accurately describe the phenomena taking place in real structures in which considerable displacements or deformations appear. Such structures are becoming increasingly common owing to the advances in materials engineering.

The influence of geometric nonlinearity connected with large displacements on the behavior of bar structures has been studied by many authors. Since the number of works dealing with this subject is very large, this survey is limited to only a few selected papers on Timoshenko beams. Pai and Plazotto in [1] used the multiple shooting method to numerically verify a nonlinear elastic cantilever bar model and the associated theoretical solutions. Mohyeddin and Fereidoon in [2] considered the large deflections of a straight prismatic shear-deformable beam resting on simple supports at both ends and subjected to a point load at its midspan. Tang et al. in [3] studied free vibration of non-uniform functionally graded beams via the Timoshenko beam theory. Capsoni et al. in [4] considered the dynamic response of a Timoshenko beam with distributed internal viscous damping. Wang and Chou in [5] analyzed the problem of a mass moving on a beam and studied the influence of the beam's non-linearity and mass on the response of the structure. Also Mamandi and Kargarnovin [6] analyzed the behavior of an isotropic prismatic Timoshenko beam loaded with a moving mass. In [7] Simsek presented an analysis of the nonlinear behavior of the Timoshenko beam under a moving harmonic load. Similar problems were studied by Guo and Zhong [8]. Using the differential quadrature method for a simply supported beam with nonslidable ends they obtained interesting results showing the influence of nonlinearity on the frequency of vibrations depending on the amplitude of the latter. Ghayesh and Balar in [9] compared two nonlinearity models describing the Timoshenko beam moving in the direction consistent with its longitudinal axis. In [10] Ghayesh and Amabili considered the stability of the moving Timoshenko beam. Ansari et al. [11] and Asghari et al. in[12] used modified nonlinear Timoshenko beam models to describe the

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behavior of nanobeams, including the loss of stability with the post-buckling stage taken into account. Aristizabal-Ochoa in [13] analyzed the lateral stability and induced bending moments, shears, and second-order deflections in columns with initial geometric imperfections and nonlinear semirigid connections subjected to axial load at both ends and lateral load at the top ends. Xia et al. in [14] studied the nonlinear bending, the post-buckling behavior and the free vibration of microbeams. They analyzed a simply supported beam with non-slidable supports. Similar problems for carbon nanotubes were solved by Yang et al. in [15]. A solution to the nonlinear vibration problem by means of the nonlinear finite element method was presented by Gunda et al. in [16]. The problem of the nonlinear vibration of the Timoshenko beam-column partially supported on the tensionless Winkler foundation was described by Sapountzakis and Kampitsis in [17]. The influence of shear deformation on the tangential displacements of the cantilever beam was described by Sapountzakis and Mokos in [18].

The analyses presented in the above papers were for prismatic beams. The common characteristic of most of the papers is that only component $(W')^2/2$ (where W represents the transverse displacement of the beam) was used to describe geometric nonlinearity.

This paper presents and compares solutions to the problem of the statics of a beam described according to Timoshenko theory for different geometric nonlinearity models. The models differ in the complexity of the nonlinear part of the strain tensor. The beam is nonprismatic and it is analyzed assuming different configurations of its support. It is shown that for certain boundary conditions the usually adopted simplified nonlinearity model (nonlinearity described by only the $(W')^2/2$ component) leads to incorrect results. Also the problem of the loss of stability of a nonprismatic bar with a preliminary geometric imperfection, loaded with an axial force is analyzed. The obtained results are compared with the results yielded by the finite element method.

In order to solve the above problems the linearization of nonlinear differential equations by means of the iteration Newton–Kantorowicz method [19] is used. The method reduces the starting equations to a sequence of approximations in the form of ordinary differential equations. The theorem put forward by Paszkowski [19], which describes the method of solving linear ordinary differential equations with variable coefficients by means of the Chebyshev series, is introduced into the above equations. The method comes down to the derivation of recurrent relations in the form of an infinite system of algebraic equations, the solution to which are the coefficients of the series expansions of the sought functions. The method is also described in the present author's paper [20]. Owing to the general character of the method, by deriving recurrent formulas for a particular problem (a given system of ordinary differential equations) one can solve the problem for different geometric and material parameters.

2. Problem formulation

The subject of the considerations is a nonprismatic Timoshenko beam with a rectilinear axis, loaded with forces $r(X)$ tangent to and forces $p(X)$ normal to the beam axis, and with bending moments $m(X)$. A schematic of the system is shown in Fig. 1.

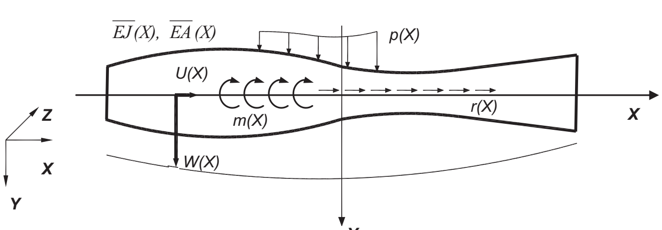

Fig. 1. Schematic of nonprismatic Timoshenko beam with rectilinear axis.

In the case of the Timoshenko beam, the total displacements in point (X, Y) are defined by formulas:

$$u_X(X,Y) = U(X) - Y\Phi(X), \quad u_Y(X,Y) = W(X) \tag{1}$$

Using relation (1) and formula $e_{ij} = e_{ji} = 1/2\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)$ one gets the following nonzero strain tensor components

$$\begin{aligned} e_{XX} &= u_{X,X} + \tfrac{1}{2}\left(\alpha\, u_{Y,X}^2 + \beta\, u_{X,X}^2\right) = U' - Y\Phi' + \tfrac{1}{2}\alpha\left[(W')^2\right] + \tfrac{1}{2}\beta\left[(U')^2 - 2U'Y\Phi' + Y^2(\Phi')^2\right] \\ e_{XY} &= e_{YX} = \tfrac{1}{2}\left(u_{X,Y} + u_{Y,X} + \gamma u_{X,X}u_{X,Y}\right) = \tfrac{1}{2}\left[-\Phi + W'\right] + \tfrac{1}{2}\gamma\left[-U'\Phi + Y\Phi'\Phi\right] \\ e_{YY} &= \tfrac{1}{2}\delta u_{X,Y}^2 = \tfrac{1}{2}\delta\Phi^2 \end{aligned} \tag{2}$$

Parameters $\alpha, \beta, \gamma, \delta$ connected with geometric nonlinearity will be used to study the influence of the individual nonlinear components on the solution of the analyzed systems. In most of the works known to the authors only the component connected with α is taken into account. As shown later in this paper, the use of only this component in the description of the systems and the omission of the other components ($\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 0$) results in considerable differences between the solutions.

The elastic strain energy (potential energy) of the system is expressed by the formula

$$U = \frac{1}{2}\int_V e_{ij}\sigma_{ij}dV \tag{3}$$

where: i, $j = X$, Y and

$$\sigma_{XX} = Ee_{XX}, \quad \sigma_{XY} = \sigma_{YX} = 2Ge_{XY} \quad \sigma_{YY} = Ee_{YY} \tag{4}$$

Having substituted relation (4) into formula (3) one gets the following formula for the elastic strain energy of the system

$$U = \frac{1}{2}\int_V E\, e_{XX}e_{XX}dV + \frac{1}{2}\int_V 2G\, e_{XY}e_{XY}dV + \frac{1}{2}\int_V 2G\, e_{YX}e_{YX}dV + \frac{1}{2}\int_V E\, e_{YY}e_{YY}dV \tag{5}$$

The work of an external load, assuming that the latter is applied to the beam axis, is defined by the formula

$$W = \int_{-a}^{+a}(pW + rU + m\Phi)dX \tag{6}$$

Using the principle of virtual work $\delta(U - W) = 0$, having introduced dimensionless variables and functions $x = X/a$, $y = Y/a$, $u = U/a$, $w = W/a$, $\varphi = \Phi$, $\overline{EI} = EI_0\, EI$, $\overline{EA} = EI_0/a^2 EA$, $\overline{EII} = EI_0\, a^2 EII$, $\overline{GA} = EI_0/a^2 GA$, $\overline{GI} = EI_0 GI$, $\overline{p} = \frac{P_0}{a}p$, $\overline{r} = \frac{P_0}{a}r$, $\overline{m} = \frac{P_0 a}{a}m$, and dimensionless parameter $n = P_0 a^2/EI_0$, where P_0, EI_0 are reference quantities, one ultimately gets the following nonlinear system of displacement equations (in order to simplify the notation the overlines in the formulas are subsequently omitted)

$$\begin{aligned} &kGAw'' + kGA'w' - kGA\varphi' - kGA'\varphi \\ &\quad + \alpha(EA'w'u' + EAw'u'' + EAw''u') + \alpha^2\left(\frac{1}{2}EA'w'^3 + \frac{3}{2}EAw'^2w''\right) \\ &\quad + \alpha\beta\left(\frac{1}{2}EA'w'u'^2 + \frac{1}{2}EAw''u'^2 + EAw'u'u'' + \frac{1}{2}EI'w'\varphi'^2\right. \\ &\quad \left. + \frac{1}{2}EIw''\varphi'^2 + EIw'\varphi'\varphi''\right) \\ &\quad + \gamma(-kGA'\varphi u' - kGA\varphi'u' - kGA\varphi u'') = n\, p \end{aligned} \tag{7}$$

$$\begin{aligned} &kGAw' + EI\varphi'' + EI'\varphi' - kGA\varphi \\ &\quad + \beta(3EI\varphi'u'' + 3EI\varphi''u' + 3EI'\varphi'u') \\ &\quad + \beta^2\left(\frac{3}{2}EI\varphi''u'^2 + \frac{3}{2}EI'\varphi'u'^2 + 3EI\varphi'u'u'' + \frac{3}{2}EII\varphi'^2\varphi'' + \frac{1}{2}EII'\varphi'^3\right) \\ &\quad + \alpha\beta\left(\frac{1}{2}EIw'^2\varphi'' + \frac{1}{2}EI'w'^2\varphi' + EI\, w''w'\varphi'\right) \\ &\quad + \gamma\,(-2kGA\varphi u' + kGAw'u') + \gamma^2(-kGA\varphi u'^2 + kGI\varphi'^2\varphi \\ &\quad + kGI\varphi''\varphi^2 + kGI'\varphi'\varphi^2) \end{aligned}$$