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Static analysis of a thick ring on a unilateral elastic foundation

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ABSTRACT

A thick ring on a unilateral elastic foundation can be used to model important applications such as non-pneumatic tires or bushing bearings. This paper presents a reduced-order compensation scheme for computing the static deformation response of a thick ring supported by a unilateral elastic foundation to an arbitrary applied force. The ring considered is an orthotropic and extensible ring that can be treated as a Timoshenko beam. The elastic foundation is a two-parameter foundation with a linear torsional stiffness but a unilateral radial stiffness whose value vanishes when compressed or tensioned. The paper first derives the deformation response for the linear foundation case for which Fourier expansion techniques can be applied to obtain an analytical solution. Then, the nonlinear unilateral foundation problem is solved via an iterative compensation scheme that identifies regions with vanishing radial stiffness and applies a compensation force to the linear foundation model to counteract the excessive foundation forces that would not be there with the unilateral foundation. This scheme avoids the need for solving the complex set of nonlinear differential equations and gives a computationally efficient tool for rapidly analyzing and designing such systems. Representative results are compared with Finite Element Analysis (FEA) results to illustrate the validity of the proposed approach.

1. Introduction

The flexible Ring on Elastic Foundation (REF) model [1,2] is a classical one that has been studied for decades. This is because of its broad and important applications such as automotive tires [3–5], railway wheels and gears [6], and others [7].

Different criteria can be used to classify the focus of existing works that analyze REFs. The simplest categories are the treatment of the static deformation problem [7,8] vs. the dynamic problem as free vibration [6] or forced vibration [2,9]. Considering the ring mechanics, the ring has been treated as a tensioned string that has direct tensile strain but no bending stiffness [10]; as an Euler–Bernoulli beam or thin ring whose plane section remains plane and always normal to the neutral axis of the ring after deformation [7,5,9]; or as a Timoshenko beam or thick ring [11] which takes the shear deformation into account by assuming that the normal of a plane section is subject to rotation. Further distinctions exist between extensional and inextensional rings. [12,1] studied the vibration problem for both a rotating thin ring and a thick ring, and pointed out that the inextensional assumptions in thick ring theories are improper because extensional coupling effects are as important as shearing effects especially for a rotating ring.

As an important component of the REF model, the treatment of the elastic foundation can be used as another criterion to classify the existing research. Numerous works, including all of the ones cited above, assume a linear and uniformly distributed stiffness for the whole elastic foundation, independent of location and deformation state. The distributed stiffness can be modeled with one parameter [7,10] where the foundation has a stiffness only in the radial direction; or with two parameters [2,9] involving both radial and torsional stiffness values; or even more parameters [6], where in addition to the radial and torsional stiffness, a stiffness associated with the distortion of the foundation due to in-plane rotation of a cross-section of the ring is included.

Although much has been gained from linear and uniform elastic foundation assumptions, not all REF problems have a perfect linear elastic foundation with uniformly distributed stiffness. Examples for nonuniform distribution include planetary gearing where tooth meshes for the ring and planets are not equally spaced [13] and tires with non-uniformity [14]. For these type of problems, [13] studied the free vibration of rings on a general elastic foundation, whose stiffness distribution can be variable circumferentially in the radial, tangential, or inclined orientation, and gave the closed-form expression for natural frequencies and vibration modes. [14] studied the natural frequencies and mode

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shapes of rings supported by a number of radial spring elements with a constant radial stiffness using modal expansion and receptance method. However, in both works, the distributions of the stiffnesses for the elastic foundations still did not change with the deformation of the ring resting on them.

A more complicated problem was invoked by considering deformation-dependent elastic foundations, such as those with unilateral stiffness whose values vanish when compressed or tensioned. The difficulty in solving this group of problems is in the fact that the compressed or tensioned region is not known in advance. It depends on the loading and consequently on the deformation. An example for the application is the non-pneumatic tire presented in [15], whose structure is a deformable shear ring supported by collapsible spokes which offer stiffness only in tension [16]. Another example is a bushing bearing, whose external sleeve can be modeled as a ring on a tensionless foundation and the internal sleeve as a ring on a collapsible foundation. Not too many works exist that deal with such rings on unilateral foundations. [17] worked on the forced response of a thin and inextensible ring on a tensionless two-parameter foundation under a time varying in-plane load. To solve the unilateral problem, an auxiliary function is used in the coefficients of the equations to track and reflect the status of the foundation. This auxiliary function makes the differential equations of the system nonlinear and difficult to solve. Furthermore, the tangential displacement of the ring is obtained from the inextensible assumption, which cannot be adopted in a more general case, such as that with extensible Timoshenko ring. [18] studied the static deformation and the contact pressure of a non-pneumatic tire resting on collapsible spokes, when it contacts against a rigid plane ground. The governing differential equations were derived only for the thick ring modeled via Timoshenko beam theory by treating the supporting force by collapsible spokes as radial distributed force which vanishes in collapsed spoke regions. The ring was divided into three regions according to the post-deformation status of the spokes (tensioned or buckled spokes) and contact status with the ground (contact region or free region). Closed-form expressions for the deformation and contact stress are given in terms of angular bounds of these three regions, which then need to be solved numerically. However, the method has limitations in two aspects: (1) For more complex loading cases, such as that with multiple forces applied at multiple locations, the number of regions into which the ring must be divided grows with the number of the load regions. Multiple unknown angular bounds would then need to be solved numerically. (2) It is difficult to extend this method to practical dynamic cases.

The present paper studies the deformation of a thick ring on an elastic two-parameter foundation where the radial stiffness is unilateral. The deformation response to an arbitrary in-plane force is considered. The ring is modeled as an orthotropic and extensible circular Timoshenko beam. As the first step, the linear foundation problem is solved analytically using Fourier expansion techniques for both the radial and tangential directions. It is then shown that the linear foundation case includes an excessive foundation force compared with that of the unilateral foundation. An iterative compensation scheme is then set up to both find the region of vanishing radial stiffness with the unilateral foundation and that of the required compensation force to counteract the excessive force predicted via the linear foundation. The method is an intuitive and efficient alternative to numerically solving the coupled and complex system of nonlinear differential equations for a flexible ring on a unilateral foundation. In addition, compared with discretization-based numerical methods such as nonlinear FEA, the proposed scheme avoids the time-consuming modeling and meshing work, which makes it attractive specially for rapid parameter studies at the design stage. Compared with the method in [18], the method proposed in this paper is capable of handling arbitrary force distributions and directions, without increasing complexity. Furthermore, since the proposed scheme is Fourier expansion-based, it can be easily extended to the dynamic cases (both forced response and dynamic contact) as we illustrate in our other work [19].

The rest of the paper is organized as follows. Section 2 restates the problem and gives the governing equations. Section 3 gives the analytical solutions for the linear foundation problem and extends them to the unilateral case. Then, in Section 4, discussions of some example results are given and compared with FEA results. Conclusions and future work are given in Section 5.

2. Statement of problem and governing equations

Fig. 1 shows a schematic of the model for a thick ring on a two-parameter elastic foundation. The ring with thickness h is assumed to have a radius R at its centroid. The width perpendicular to the plane of the ring is b. The uniformly distributed radial and tangential stiffnesses are assumed to be K_r and K_θ, respectively. They have units of stiffness per radian. For a linear elastic foundation, the distributed radial stiffness K_r is constant. However, for the unilateral elastic foundation, the radial stiffness vanishes when the elastic foundation is tensioned or compressed. A polar coordinate system with origin located at the ring center is adopted. The center of the ring is fixed and friction is neglected.

The radial and tangential displacements at the centroid are assumed to be $u_r(R,\theta)$ and $u_\theta(R,\theta)$, respectively. Following [20], the cross-section of the ring is assumed to have a rotation $\phi(R,\theta)$ at the centroid at circumferential position θ and keeps its plane after deformation. Then, the radial and tangential displacements at an arbitrary point on the ring with radius r and circumferential position θ, $u_r(r,\theta)$ and $u_\theta(r,\theta)$, can be represented by

$$u_r(r,\theta) = u_r(R,\theta)$$
$$u_\theta(r,\theta) = u_\theta(R,\theta) + (r-R)\phi(R,\theta) \quad (1)$$

The strain–displacement relationships in polar coordinates are [21]

$$\epsilon_{rr}(r,\theta) = \frac{\partial}{\partial r}u_r(r,\theta)$$
$$\epsilon_{\theta\theta}(r,\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}u_\theta(r,\theta) + \frac{1}{r}u_r(r,\theta)$$
$$\gamma_{r\theta}(r,\theta) = \frac{1}{r}\frac{\partial}{\partial\theta}u_r(r,\theta) + \frac{\partial}{\partial r}u_\theta(r,\theta) - \frac{1}{r}u_\theta(r,\theta) \quad (2)$$

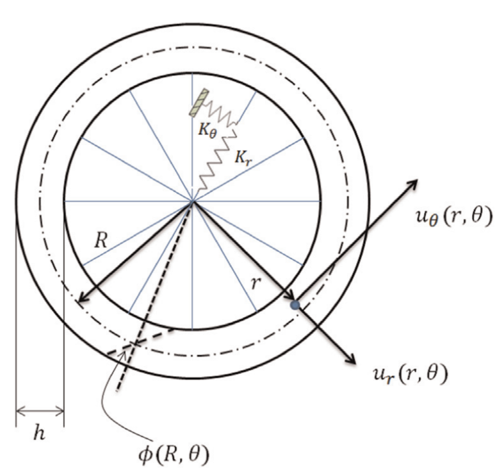

Fig. 1. Thick ring on a two-parameter elastic foundation.