International Journal of Non-Linear Mechanics 81 (2016) 8–18

Contents lists available at ScienceDirect

International Journal of Non-Linear Mechanics

journal homepage: www.elsevier.com/locate/nlm

Asymptotic limits and wrinkling patterns in a pressurised shallow spherical cap

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ARTICLE INFO

Article history:
Received 19 October 2015
Received in revised form
1 December 2015
Accepted 2 December 2015
Available online 12 December 2015

Keywords:
Shallow shells
Boundary-value problems
Wrinkling
Asymptotic methods

ABSTRACT

The asymmetric bifurcation problem for a shallow spherical cap is examined. The applied pressure can act either external or internal to the cap and both cases are treated here. Assuming a non-linear axisymmetric basic state, the linearised bifurcation equations for the pressurised shell are investigated in the limit when the thickness of the cap is much less than the maximum rise of the shell mid-surface. Within this regime the wrinkling patterns in both cases are confined to a narrow zone near the edge of the shell, making it possible to solve asymptotically the corresponding equations and derive analytical predictions for both the critical pressure and the corresponding number of wrinkles. Some comparisons with direct numerical simulations are included as well.

1. Introduction

Interest in the buckling of an externally pressurised shallow spherical cap was originally motivated by research into the behaviour of a complete spherical shell subjected to the same type of loading. Experimental evidence indicates that the full shell deforms and is susceptible to a mode characterised by a large number of adjacent circular dimples that cover the entire surface of the sphere [1], thereby giving an appearance not dissimilar to that of a golf ball. Since each of the buckles subtends only a small solid angle at the centre of the sphere, it was a small step to conclude that an understanding of the shallow cap problem with a clamped edge might shed light on the behaviour of spherical shells. Of course, this analogy between the two configurations is only partially valid, but the buckling and post-buckling behaviours of spherical caps have remained a classic topic and received a considerable amount of attention in the literature (e.g., see the recent survey [2] and the references therein). Much of this interest can be traced back to the widespread use of spherical structural elements, with geometries ranging from hemispherical to reasonably shallow. For instance, shallow and deep spherical caps are found extensively in powered submersibles [3], while in the nuclear power industry they are used in pressure safety devices [4,5]. Very recently shallow spherical caps have found increasing applicability in the design of transducers (e.g., [6]).

When subjected to a uniform *external* pressure field (i.e., directed towards the centre of curvature of the cap), a shallow spherical cap can display two distinct modes of deformation, as illustrated in the schematic Fig. 1. So-called snap-through buckling, as illustrated in the left window, is characterised by a sudden reversal of curvature over a small region near the apex and, typically, involves large displacements. This type of instability is the more studied case because the deformation experienced by the shell is axisymmetric and the governing equations are ordinary rather than partial differential equations. Detailed historical surveys regarding the experimental development of this problem can be found in the books by Bushnell [1] or Singer et al. [7] while theoretical aspects are reviewed by Shilkrut [8], with additional excellent accounts being found in [9,10].

Snap-through buckling is usually associated with the presence of a limit point on the load–(apex) displacement curve for the cap, although the instability is sometimes triggered before that point is reached. Loss of stability through this route only occurs over a relatively narrow range of values of the height-to-thickness ratio of the cap. If this ratio is very small then no instability occurs, whereas for larger values the shell tends to experience *azimuthal buckling* or *wrinkling*, as illustrated in the right window of Fig. 1. If the aforementioned ratio is large ($\simeq 15$ or greater) then the wrinkles tend to concentrate near the shell edge. The governing equations describing this type of bifurcations can still be reduced to ordinary differential equations due to the regular nature of the

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http://dx.doi.org/10.1016/j.ijnonlinmec.2015.12.004
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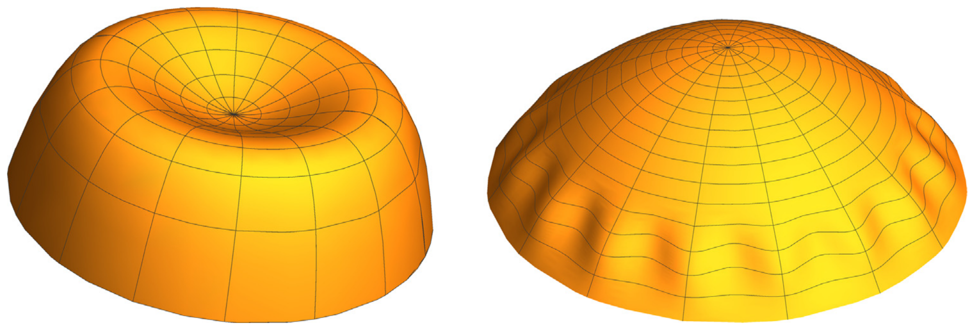

Fig. 1. An illustration of the two principal modes of instability experienced by an externally pressurised spherical shell: *snap-through buckling* (left), and *asymmetric wrinkling* (right); the deformation modes are not to scale.

wrinkling pattern, but they are of a much higher order than those for snap-through buckling.

Weinitschke [11] was the first to systematically calculate the critical pressures that trigger asymmetric buckling in shallow spherical caps. He adopted power series representations for both the non-linear basic state and the bifurcation equations, but the values predicted by his approach turned out to be quite low compared with those seen in practice. Huang [12] revisited the same problem but replaced the power series with a finite-differences numerical scheme that allowed more accurate results to be derived. Archer and Famili [13] confirmed the improved accuracy of Huang's results by using an alternative approach to buckling based on dynamical bifurcation equations. This involved examining the asymmetric frequency spectrum corresponding to small free vibrations of the shell in the neighbourhood of the non-linear axisymmetric equilibrium state. The buckling mode and corresponding critical load were deduced by observing the vanishing of a particular frequency. These authors also developed a load-perturbation approach [14] to compute numerically the incipient stage for the asymmetric post-buckling response. Fitch [15] modified Huang's numerical solution to account for the problem when a vertical load concentrated at the apex of the cap is applied, and concluded that the bifurcation pattern is characterised by between three and five circumferential waves. This continues to be the case even for large height-to-thickness ratios but in this limit the deformation is confined near the apex rather than the edge of the shell. He also used a Lyapunov–Schmidt scheme to compute the initial post-buckling behaviour and established that the bifurcation is super-critical, with the consequence that the cap under a concentrated load is an imperfection-insensitive structure.

The related situation when the shallow spherical cap is subjected to a pressure acting in the opposite direction, that is from the inside of the cap, has received considerably less attention. This configuration, which we shall henceforth refer to as the *internally pressurised shell*, was studied numerically by Shilkrut [8,16] who seems to have been the first to point out that these structures undergo asymmetric buckling that is sometimes localised near the rim of the shell. Since most of the numerical information in his book [8] is presented in terms of the so-called 'deformation maps' it is somewhat arguable as to the extent to which the analogy between the externally and internally pressurised shallow spherical caps can be pursued. One aim of our present study is to throw some more light on the possible connections between the two problems by conducting an in-depth numerical exploration, and thereby establish the asymptotic structure of the corresponding bifurcation equations when the height-to-thickness ratio is large.

Additional motivation is provided by some recent related work [18,19] in which we explored the edge-wrinkling of pressurised thin elastic plates. As a flat circular plate may be thought of in terms of a degenerate spherical cap, it is of interest to gain some further understanding of the similarities and differences between the two configurations in the context of asymmetric bifurcations. The transition from shell to plate seems to have received scant attention in the literature although a noteworthy exception is the intriguing work [20]. In this study a particular type of spherical cap was pulled at the rim (see [21] for details) but deformations were restricted to axisymmetric forms.

To keep the paper reasonably self-contained, we commence our study by reviewing the key equations of interest and identify some suitable dimensionless parameters. As shown in Sections 2 and 3 the externally and internally pressurised shallow shell problems can be described by almost the same differential equations, with the only discrepancy arising from a sign change that indicates the direction of the applied pressure (see comments immediately after Eq. (2.5)). The boundary conditions for the two problems are somewhat different and these are spelled out in detail in those introductory sections. We proceed in Section 4 to conduct a numerical investigation of the dependence of the lowest critical wrinkling pressure on various quantities of interest. Further numerical work provides evidence as to the mechanisms responsible for the localisation of the wrinkling patterns, and samples of representative eigenmodes obtained. Guided by the numerical evidence of Section 4, we then discuss the asymptotic structure of the externally pressurised shallow cap in Section 5 and compare with the results of the numerical simulations. A parallel asymptotic investigation is carried out in Section 6 for the internally pressurised shell. The paper concludes with a discussion of our main findings and some remarks on possible extensions.

2. The basic state

We consider a shallow spherical cap of uniform thickness $h>0$, whose middle surface can be represented by the elliptic paraboloid $z=H\left[1-(r/a)^2\right]$, where H is the rise of the middle surface at the centre, and a denotes the base radius. The geometry of the configuration is sketched in Fig. 2; with our chosen dimensions the curvature radius of the shell is $R\equiv a^2/(2H)$ and the cap is shallow so long as $0<h/R\ll 1$. The linearly isotropic elastic material of the spherical cap is described by Young's modulus E and the Poisson ratio ν. Our two problems of interest are distinguished by the direction of application of the uniform pressure. In the first case, the shell experiences an external pressure P normal to its surface (see Fig. 3(a)), and we shall refer to this problem as $\mathcal{C}_{\rm sph}^{(-)}$. The second situation, as depicted in Fig. 3(b) and denoted by $\mathcal{C}_{\rm sph}^{(+)}$, is the far less studied problem of an outwardly pressurised spherical cap. Finally, it will be further assumed that suitable edge restraints are applied, and this will be made more precise shortly.

The starting point for setting up the relevant bifurcation problem is the well-known Donnell–Mushtari–Vlasov (DMV) shallow