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Supercritical and subcritical buckling bifurcations for a compressible hyperelastic slab subjected to compression

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A B S T R A C T

We study the buckling bifurcation of a compressible hyperelastic slab under compression with sliding–sliding end conditions. The combined series-asymptotic expansions method is used to derive the simplified model equations. Linear bifurcation analysis yields the critical stress value of buckling, which gives a non-linear correction to the classical Euler buckling formula. The correction is due to the geometrical non-linearities coupled with the material non-linearities. Then through non-linear bifurcation analysis, the approximate analytical solutions for the post-buckling deformations are obtained. The amplitude of buckling is expressed explicitly in terms of the aspect ratio, the incremental dimensionless engineering stress, the mode of buckling and the material constants. Most importantly, we find that both supercritical and subcritical buckling could occur for compressible materials. The bifurcation type depends on the material constants, the geometry of the slab and the mode numbers.

1. Introduction

In this paper, we study the buckling and the post-buckling of a compressible slab subjected to compressions with sliding–sliding end conditions. We intend to obtain the critical stress of buckling and the post-buckling solutions.

This problem is related to the celebrated Euler buckling formula which gives the critical stress value of bars under compression (see [1]). In Euler buckling theory, all the buckling modes are supercritical (see [2]). Euler buckling theory can predict the critical stress values very well for materials with small deformation under compression. Recently, some results were obtained for the non-linear corrections of the Euler formula. For the incompressible cylindrical tubes with sliding–sliding end conditions, Goriely et al. [3] obtained a non-linear correction formula. For an incompressible slab with sliding–sliding end conditions, Dai and Wang [4] obtained a corrected buckling formula by using model equations reduced from plane strain non-linear elasticity. For a compressible slender column with sliding–sliding end conditions, Pascalis et al. [5] obtained the non-linear correction by using incremental theory. That correction is due to the geometrical non-linearities coupled with the material non-linearities (the third-order elastic constants). For a compressible slab with clamped–clamped end conditions, Dai and Wang [6] obtained the critical stress values and compared these values with those obtained from the Euler formula numerically. Based on the prestressed problem, Dai et al. [7] provided a criterion to predict the supercritical or subcritical buckling for a two-dimensional slab composed of the compressible neo-Hookean material. However, this criterion is complicated and was not given explicitly. So it is of interest to find how the material constants, the geometry of the slab and the mode numbers affect the bifurcation type for general compressible materials. We intend to find a simpler criterion here.

There are some theoretical analyses of this type of compression problems in the literature (see [8–13], etc.). From linear theories, one can only obtain the critical loads and eigenfunctions. Then there are very few results about the analytical post-bifurcation solutions except recently [4,7,14]. These results are obtained by using the method of combined series-asymptotic expansions which was developed in [15–17], etc. In [4,7], post-buckling behaviors are studied for compression of incompressible and compressible slabs, respectively. In [14], post-bifurcation solutions are obtained for compression of a compressible hyperelastic tube. In the post-bifurcation analysis, the authors found pitchfork and octopus bifurcations for thin tubes, which cannot be observed in the linear bifurcation analysis. In this paper, not only we will obtain the critical stress value of buckling, but we will also determine the bifurcation types of buckling and obtain the post-buckling solutions.

The structure of this paper is the following. In Section 2, we give the mathematical formulation of the problem. In Section 3, we introduce the derivations of the model equations. In Section 4, we study the bifurcations of the model equations under sliding–sliding

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end conditions. Linear bifurcation analysis yields the critical stress values of buckling. Through non-linear bifurcation analysis, we obtain the post-buckling solutions analytically and compare the analytical solutions with the numerical solutions. Finally, we give some conclusions in Section 5.

2. Mathematical formulation of the problem

In this section, we present the mathematical formulation of the problem. The compression of a two-dimensional slab is regarded here as a plane strain problem. As shown in Fig. 1, the slab is compressed between two lubricated rigid bodies at $X=0, l$, and is not allowed to deform in the Z-direction out of plane. We consider the case that the slab is composed of a compressible isotropic hyperelastic material with the strain energy function Φ. Let the length of the rectangle be l and the thickness be $2a$ and let (X, Y, Z) and (x, y, z) denote the Cartesian coordinates of a material point in the reference and current configurations, respectively. The axial and lateral displacements are denoted by

$$U(X,Y)=x-X, \quad V(X,Y)=y-Y, \tag{1}$$

respectively. Then the deformation gradient tensor $\mathbf{F}$ is

$$\mathbf{F}=(U_X+1)\mathbf{e}_x\otimes\mathbf{E}_X+U_Y\mathbf{e}_x\otimes\mathbf{E}_Y+V_X\mathbf{e}_y\otimes\mathbf{E}_X+(V_Y+1)\mathbf{e}_y\otimes\mathbf{E}_Y+\mathbf{e}_z\otimes\mathbf{E}_Z, \tag{2}$$

where $\mathbf{E}_X, \mathbf{E}_Y, \mathbf{E}_Z$ and $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ represent the orthonormal bases in the reference and current configurations, respectively.

We denote by I_i $(i=1,2,3)$ as the three principal invariants of the left Cauchy–Green deformation tensor $\mathbf{B}=\mathbf{F}\mathbf{F}^{\mathbf{T}}$. Since we are considering a plane strain problem, the strain energy function Φ can be seen as a function of I_1 and I_2 only, since $I_3=I_2-I_1+1$ (see [18]).

The first Piola–Kirchhoff stress tensor $\mathbf{\Sigma}$ containing terms up to the third-order material non-linearity for an arbitrary strain energy function can be calculated by a formula provided in [19]

$$\Sigma_{ij}=a^1_{jilk}d_{kl}+\tfrac{1}{2}a^2_{jilknm}d_{kl}d_{mn}+\tfrac{1}{6}a^3_{jilknmqp}d_{kl}d_{mn}d_{pq}+O(|d_{ij}|^4), \tag{3}$$

where (d_{ij}) is the displacement gradient tensor $\mathbf{d}=\mathbf{F}-\mathbf{I}$, a^1_{jilk}, a^2_{jilknm} and $a^3_{jilknmqp}$ are elastic moduli defined by

$$\begin{cases} a^1_{jilk}=\left.\dfrac{\partial^2\Phi}{\partial F_{ij}\partial F_{kl}}\right|_{\mathbf{F}=\mathbf{I}}, \quad a^2_{jilknm}=\left.\dfrac{\partial^3\Phi}{\partial F_{ij}\partial F_{kl}\partial F_{mn}}\right|_{\mathbf{F}=\mathbf{I}}, \\ a^3_{jilknmts}=\left.\dfrac{\partial^4\Phi}{\partial F_{ij}\partial F_{kl}\partial F_{mn}\partial F_{st}}\right|_{\mathbf{F}=\mathbf{I}}. \end{cases} \tag{4}$$

Hereafter we use Φ_{ij} to denote the i-th order derivative of Φ with respect to I_1 and j-th order derivative of Φ with respect to I_2 evaluated at $\mathbf{F}=\mathbf{I}$. Then the elastic moduli in (4) can be expressed in terms of Φ_{ij}.

Here we study a static problem. The field equations (neglecting the body force) are

$$\frac{\partial\Sigma_{xX}}{\partial X}+\frac{\partial\Sigma_{xY}}{\partial Y}=0, \tag{5}$$

$$\frac{\partial\Sigma_{yX}}{\partial X}+\frac{\partial\Sigma_{yY}}{\partial Y}=0. \tag{6}$$

Substituting the above stress components into (5) and (6), we obtain two complicated coupled non-linear partial differential equations.

Since there is no distributive loading on the lateral boundaries, the stress components Σ_{xY} and Σ_{yY} should vanish there. We have thus the following traction-free boundary conditions:

$$\Sigma_{xY}=0 \quad \text{at } Y=\pm a, \tag{7}$$

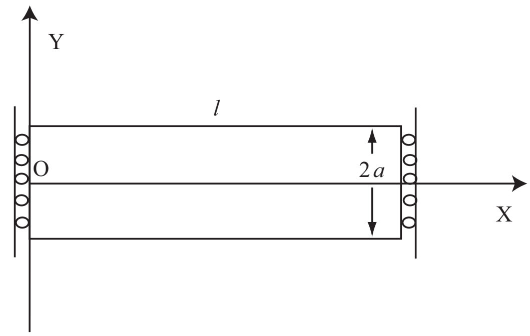

Fig. 1. Sketch map of a compressed slab.

$$\Sigma_{yY}=0 \quad \text{at } Y=\pm a. \tag{8}$$

We will study the bifurcations of the field equations (5) and (6) under traction-free boundary conditions (7) and (8) and sliding–sliding end conditions.

3. Derivations of the model equations

In this section, we follow the combined series-asymptotic method (see [15–17]) to derive the simplified model equations for compressions of a compressible slab. First, through the following scalings:

$$U=hu, \quad V=hv, \quad X=\bar{x}l, \quad Y=\bar{y}l, \quad \varepsilon=\frac{h}{l}, \quad \xi=\frac{a^2}{l^2}, \tag{9}$$

the system (5) and (6) and the conditions (7) and (8) can be non-dimensionalized. The parameter h is the characteristic axial displacement which can be regarded as the reduction of the distance between two ends, and ε will be treated as a small parameter since here the deformation is considered to be small. The over bars of $\bar{x}$ and $\bar{y}$ are dropped hereafter for convenience.

We assume that the slenderness of the slab is small, so that ξ is a small parameter. And then y is a small variable and $u(x,y)$ and $v(x,y)$ have the following series expansions in the neighborhood of $y=0$:

$$u(x,y)=u_0(x)+y^2u_2(x)+\cdots+\delta y(u_1(x)+y^2u_3(x)+\cdots), \tag{10}$$

$$v(x,y)=\delta(v_0(x)+y^2v_2(x)+\cdots)+y(v_1(x)+y^2v_3(x)+\cdots), \tag{11}$$

where δ is a parameter and δh represents the characteristic deflection of the central axis.

Substituting (10) and (11) into the traction-free boundary conditions, we arrive at four complicated equations containing ten unknowns $(u_0,\ldots,u_4,v_0,\ldots,v_4)$:

$$\mathcal{D}_1(u_0,u_1,u_2,u_3,v_0,v_1,v_2,v_3)+O(\xi^2,\varepsilon\xi,\varepsilon^2\xi\delta^2)=0, \tag{12}$$

$$\mathcal{D}_2(u_0,u_2,u_4,v_1,v_3)+O(\xi^2,\varepsilon\xi,\varepsilon\xi\delta^2)=0, \tag{13}$$

$$\mathcal{D}_3(u_0,u_2,v_1,v_3)+O(\xi^2,\varepsilon\xi,\varepsilon\xi\delta^2)=0, \tag{14}$$

$$\mathcal{D}_4(u_0,u_1,u_2,u_3,v_0,v_1,v_2,v_3,v_4)+O(\xi^2,\varepsilon\xi,\varepsilon^2\xi\delta^2)=0, \tag{15}$$

where $\mathcal{D}_i$ $(i=1,2,3,4)$ are operators of the corresponding functions.

To have a closed system, we substitute (10) and (11) into the field equations (5) and (6) and the left-hand sides become two infinite series in y. From the coefficients of y^0, y^1 and y^2, we get six equations with the same ten unknowns, which are very long and not written out explicitly for brevity.

Then the field equations and the traction-free boundary conditions are changed into a one-dimensional system of ten differential