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Steady flow through a slender tapered vessel with a partially permeable wall and closed end

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ABSTRACT

We study the flow of a viscous fluid through a slender tapered tube whose radius may reduce to zero. The vessel is closed at the end, so that the flow is made possible owing to the fact that a portion of the tube wall is permeable. The smallness of the tube aspect ratio is exploited using an upscaling technique leading to a degenerate differential equation for pressure. Solutions are found either in explicit form or as power series expansions. This class of flows may represent, though in a largely approximated way, the blood flow though a coronary artery.

1. Introduction

Incompressible flows through tapered tubes have been studied extensively, particularly in the framework of blood flow in stenosed vessels. We quote the early papers [11,10] on Newtonian flows and [13,12] on non-Newtonian flows. The experiments in [17] on blood flow through tapered tubes stimulated various theoretical investigations. A comparison between Newtonian and non-Newtonian flows in tapered tubes has been performed in [8], where the relevant literature has been shortly reviewed. Recent papers on blood flow (with various rheological structure) in arteries with stenosis are [2,9,14,5].

The biological background of the present study is still in the framework of blood flow, but with the substantial difference that the tapered vessel is closed at the end and the flow is exclusively due to the permeability of part of the wall. This is a physiological rather than pathological condition when we consider coronary arteries, which narrow to the end and provide with blood the capillary network of the myocardium through multiple branching (we refer to the extensive paper by Dunker and Bache [4]). In a rough scheme one could describe the blood outflow as due to a partial permeability of the vessel wall, and the flow occurring in the permeable membrane as driven by the pressure difference between the inner vessel and the external region (whose pressure is assumed to be uniform and constant). That said, it is quite clear that the model we are going to discuss cannot be taken as a real description of the blood flow in coronary arteries, for many reasons, though at least some features of the coronary flow during systole are captured. In view of such a correspondence we will select all characteristic quantities in such a way that they are relevant to coronary flow (see, e.g., [7]).

For the readers interested in the literature on modeling coronary flow we quote [1,7,15].

Despite the many simplifications introduced, the fluid dynamical problem we are going to study is quite complicated. Exploiting the smallness of the aspect ratio (maximal radius/length), we use an upscaling technique, in conjunction with the assumption of steady flow, to obtain a boundary value problem for a degenerate differential equation for pressure, which is solved in some case in explicit form, or more generally via classical power series expansions. Various geometrical shapes of the tube will be considered which give rise to different qualitative behavior of the flow and also require a different mathematical approach. Numerical examples are provided. The mathematical technique employed generalizes the one adopted in [6] for the study of flows in cylindrical tubes with porous walls. Our analysis reveals that the behavior of pressure near the tip is critically dependent on the tip shape.

2. The mathematical model

We consider a vessel with a thinning circular cross section. We denote by z^*, and r^* the longitudinal and radial coordinates, and by θ, $\theta\in(-\pi,\pi]$, the angular variable. The vessel's length is[1] L^*, and

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[1] Throughout this paper the superscript "*" denotes dimensional variables.

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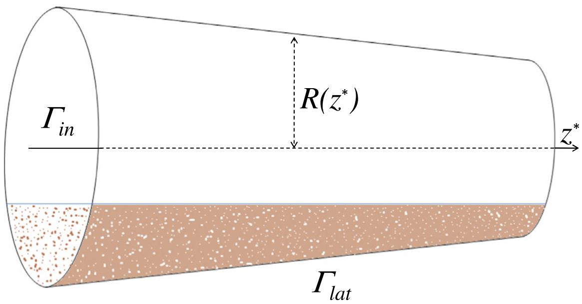

Fig. 1. The strip Γ_{lat} is the permeable part of the tapered vessel lateral boundary, while Γ_{in} is the inlet. Actually, in this paper we analyze the case $R(z^*)\to 0$ as $z^*\to L^*$.

its internal radius is $R^*R(z^*)$, where $R(z)$, is dimensionless, decreasing, continuously differentiable and $0\le R(z^*)\le 1$. We assume that $\varepsilon=\frac{R^*}{L^*}\ll 1$.

A Newtonian incompressible fluid, of constant density ρ^* and viscosity μ^*, enters the tube through the inlet surface Γ_{in} at $z^*=0$, and leaks through a portion of the lateral surface

$$\Gamma_{lat}=\{0\le x^*\le L^*,\ r^*=R^*R(z^*),\ \theta_o<\theta<\theta_1\},$$

acting as a porous membrane characterized by a ratio[2] K^* between membrane permeability and membrane thickness (see Fig. 1). Such a lateral flow is assumed to be Darcian, i.e. proportional, via K^*, to the transmural pressure difference.

We denote by $\boldsymbol{v}^*=u^*\boldsymbol{e}_z+v^*\boldsymbol{e}_r+w^*\boldsymbol{e}_\theta$, the fluid velocity within the tube and by V_c^*, the characteristics longitudinal fluid velocity (that can be evaluated once we have an estimate for the discharge). A basic step in our investigation is to introduce a double rescaling for the longitudinal and the transverse components

$$u=\frac{u^*}{V_c^*},\quad v=\frac{v^*}{\varepsilon V_c^*},\quad \text{and}\quad w=\frac{w^*}{\varepsilon V_c^*},$$

so that $\boldsymbol{v}^*=V_c^*(u\boldsymbol{e}_z+\varepsilon v\boldsymbol{e}_r+\varepsilon w\boldsymbol{e}_\theta)$. Likewise we define $x=x^*/L^*$ and $r=r^*/\varepsilon L^*$. Concerning the fluid pressure, $p^*=p^*(x^*,r^*,t^*)$, measured with reference to the external pressure (which we have set equal to 0), we rescale it by p^*_{ref} which we select imposing

$$\underbrace{\pi R^{*2}V_c^*}_{\text{inlet flow}}=\underbrace{(\theta_1-\theta_o)R^*L^*\frac{K^*}{\mu^*}p^*_{ref}}_{\text{lateral flow}},$$

namely

$$p^*_{ref}=\frac{\varepsilon\pi\mu^*V_c^*}{(\theta_1-\theta_o)K^*}. \tag{1}$$

The fluid mechanical incompressibility is expressed by

$$\nabla^*\cdot\boldsymbol{v}^*=0. \tag{2}$$

The non-dimensional version of (2) is

$$\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial}{\partial r}(r\,v)+\frac{1}{r}\frac{\partial w}{\partial\theta}=0. \tag{3}$$

Neglecting gravity, the steady flow is governed by the Stokes equation

$$-\nabla^*p^*+\mu^*\Delta^*\overrightarrow{v}^*=0. \tag{4}$$

The dimensionless form of (4) involves the dimensionless parameter

$$\beta=\frac{(\theta_1-\theta_o)L^{*2}K^*}{\pi R^{*3}}=\frac{(\theta_1-\theta_o)}{\pi}\frac{K^*}{\varepsilon^2R^*}. \tag{5}$$

[2] K^* is dimensionally a length.

Assuming $\beta=\mathcal{O}(1)$, the dimensionless form of (4), in which the $\mathcal{O}(\varepsilon)$ terms are neglected (lubrication approximation), is

$$0=-\frac{1}{\beta}\frac{\partial p}{\partial z}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{6}$$

$$0=\frac{\partial p}{\partial r}, \tag{7}$$

$$0=\frac{\partial p}{\partial\theta}. \tag{8}$$

It is interesting to remark that when $\beta\le\mathcal{O}(\varepsilon)$, a condition of negligible permeability, the zero order approximation of the pressure gives $\frac{\partial p}{\partial r}=\frac{\partial p}{\partial\theta}=\frac{\partial p}{\partial z}=0$, namely a uniform pressure field not generating any flow.

The boundary conditions that we impose on the vessel lateral surface are

$$\boldsymbol{u}^*\cdot\boldsymbol{t}_\theta=\boldsymbol{u}^*\cdot\boldsymbol{t}_z=0,\quad\text{no-slip}, \tag{9}$$

$$\boldsymbol{u}^*\cdot\boldsymbol{n}=K^*p^*\chi_{[\theta_o,\theta_1]}(\theta),\quad\text{outflux}, \tag{10}$$

where:

- $\boldsymbol{t}_\theta$, $\boldsymbol{t}_z$, and $\boldsymbol{n}$ are the tangent and normal unit vectors to the vessel surface, respectively.
- $\chi(\theta)=1$ for $\theta\in[\theta_o,\theta_1]$, and $\chi(\theta)=0$ otherwise. Condition (10) means that the vessel surface is permeable only for $\theta_o\le\theta\le\theta_1$, and the flow is driven by the pressure difference with respect to a far field.

We may have an estimate of β and K^* in a specific case on the basis of the data reported in Table 1, compatible with the coronary flux during systole.

Assuming $\frac{\partial R}{\partial z}=\mathcal{O}(1)$, the dimensionless form of (9) and (10), approximated up to $\mathcal{O}(\varepsilon)$ terms, is

$$u|_{r=R}=0, \tag{11}$$

$$w|_{r=R}=0, \tag{12}$$

$$v|_{r=R}=\frac{\pi}{\theta_1-\theta_o}p\chi_{[\theta_o,\theta_1]}(\theta). \tag{13}$$

Now, taking (11)–(13) into account, we integrate Eq. (3) over the whole vessel section

$$R(z)\underbrace{\int_{-\pi}^{\pi}v|_{r=R}\,d\theta}_{\pi p}+\int_0^{R(z)}\int_{-\pi}^{\pi}r\frac{\partial u}{\partial z}\,dr\,d\theta=0. \tag{14}$$

Next, recalling (7) and (8), we conclude that $p=p(z,t)$. So, considering Eq. (6), supplemented with boundary conditions (11) and with the symmetry condition at $r=0$, we obtain the following

Table 1
Typical geometrical and physical values of coronary flow [7].

Quantity	Typical value	Units
R^*	2×10^{-1}	cm
L^*	10	cm
ε	2×10^{-2}	
V_c^*	~10	cm/s
ρ^*	1	gr/cm^3
μ^*	1.2	cP
p^*_{ref}	80–100	mmHg
K^*	10^{-8}–10^{-7}	cm
$(\theta_1-\theta_0)$	π	
β	10^{-2}–10^{-1}	