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Steady state response analysis for fractional dynamic systems based on memory-free principle and harmonic balancing

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ABSTRACT

A semi-analytic approach is proposed to analyze steady state responses of dynamic systems containing fractional derivatives. A major purpose is to efficiently combine the harmonic balancing (HB) technique and Yuan–Agrawal (YA) memory-free principle. As steady solutions being expressed by truncated Fourier series, a simple yet efficient way is suggested based on the YA principle to explicitly separate the Caputo fractional derivative as periodic and decaying non-periodic parts. Neglecting the decaying terms and applying HB procedures result into a set of algebraic equations in the Fourier coefficients. The linear algebraic equations are solved exactly for linear systems, and the non-linear ones are solved by Newton–Raphson plus arc-length continuation algorithm for non-linear problems. Both periodic and triple-periodic solutions obtained by the presented method are in excellent agreement with those by either predictor–corrector (PC) or YA method. Importantly, the presented method is capable of detecting both stable and unstable periodic solutions, whereas time-stepping integration techniques such as YA and PC can only track stable ones. Together with the Floquet theory, therefore, the presented method allows us to address the bifurcations in detail of the steady responses of fractional Duffing oscillator. Symmetry breakings and cyclic-fold bifurcations are found and discussed for both periodic and triple-periodic solutions.

1. Introduction

It has been reported that in many cases fractional derivatives provide a better model of a system than integer order derivatives do [1–3]. Recent decades have witnessed the increasing research interests in fractional dynamic systems in various fields of science and engineering. Meanwhile, it has become more and more urgent for researchers to develop effective solution techniques to deal with systems containing fractional derivatives [4–7].

Basically, there are two categories of solution approaches for solving fractional dynamic systems. One is based on time-stepping integration such as predictor–corrector (PC) algorithm [4], and the other refers to analytic [7] or semi-analytic methods such as frequency domain analysis approaches [8]. As we know, the fractional derivative exhibits a global feature because it is expressed by a convolution over the whole solution domain. Direct numerical simulation is very time-consuming owing to the repeated evaluation of the convolution over intervals [0, t] that grow like t. In addition, it requires storing the history responses because of the repeated usage in processing the convolution. For these reasons, long-term simulation has to be realized at the expense of dramatically increasing computational efforts. Generally, the computational cost increases in proportional to n^2 with n as the number of integration steps [4]. The cost can be reduced to $n\log(n)$ by using the short memory principle [9] or some other strategies [10]. According to the short memory principle, the fractional derivative is only integrated over a fixed period of recent history by neglecting the tail of the convolution.

To further release the restriction of huge storage of history responses and tedious computation of the convolutions, researchers proposed memory-free approaches such as the numerical scheme initiated by Yuan and Agrawal (YA method) [11], and that by Atanackovic and Stankovic [12]. The key of the YA method is to transform the fractional derivative into an improper integral with time as a parameter, and to approximate the integral in the time-domain using Gauss–Laguerre integration. By these procedures, a set of ordinary differential equations will be deduced. Without any fractional derivatives, the deduced systems can then be directly solved by some standard time-stepping algorithms, such as the Newmark method [13] or the trapezoidal rule [14]. Schmidt and Gual reported that, the YA approach is incapable of approximating the creep response of the massless Kelvin–Voigt model [14]. For this issue, Agrawal modified this method to better approximate the creep response [15]. Based on the YA approach, several improvements were further proposed for its better performance [16,17].

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Steady state responses, if there are any, are of fundamental significance to the researches in dynamic systems. Very often, amplitude–frequency curves are obtained by analyzing steady state responses such as periodic or limit cycle oscillations. Both linear and non-linear fractional dynamic systems subjected to harmonic excitations usually exhibit periodic responses [18,19], and some self-excited systems give rise to limit cycle oscillations [20]. By time-stepping numerical approaches, steady responses are usually truncated from long-term simulations when the influence of initial conditions has faded enough. As mentioned above, long-term simulation is usually a severe computational obstacle. Another weakness is time-stepping approaches is incapable of detecting unstable solutions.

It may be a better choice to implement frequency domain methods because they can solve steady state responses directly. To the best of our knowledge, frequency analysis methods have not been well-developed to investigate steady state responses of fractional dynamic systems. The reasons probably lie in the facts that the fractional derivative of a periodic function is usually non-periodic [21], and it is cumbersome to process the fractional derivative in the frequency domain [22]. Even though, researchers have still devoted to developing frequency domain methods for the purpose of efficiently getting steady solutions of fractional dynamic systems. For example, Leung et al. [8] and Xiao et al. [23] applied the residue harmonic balance method to solve the fractional van der Pol type oscillators. Xie and Lin [24] analyzed the van der Pol oscillator with small fractional damping using the method of two-scales expansion. Shen et al. [25] implemented the averaging method to study the primary resonance of Duffing oscillator with fractional derivative.

The key computational obstacle confronted in frequency domain analysis lies in calculating the fractional derivative of periodic function. In many cases, the calculations were manipulated directly but inefficiently since the fractional derivative functions are not evident [21,22]. In the YA method, the fractional derivative is transformed from a convolution into a parametric improper integral. Here, we call this transformation as the YA memory-free principle for convenience. It will be shown later, this parametric integral provides us a convenient way to compute the fractional derivative of periodic function. Motivated by this convenience, we combine the YA principle and harmonic balance (HB) procedure to propose an analytical or semi-analytical approach for investigating the steady state responses of fractional dynamic systems. The HB method is to express a periodic solution as truncated Fourier series, and to determine the Fourier coefficients by solving a set of algebraic equations deduced by equating the coefficients of the cosine and sine functions to zeroes for the considered equations [26].

Both linear and non-linear systems are presented as numerical examples. The results agree excellently with those obtained by the PC algorithm [4] and by the YA method [11], respectively. Since the solutions by the presented method are expressed as truncated Fourier series, their stabilities can be judged by the Floquet theories. It is shown that, the presented method is capable of solving both stable and unstable periodic solutions. Due to this merit, moreover, we find and analyze the bifurcations in detail of both periodic and triple-periodic solutions. On the other hand, the time domain solution provided by time-stepping numerical methods is intuitively considered to be stable [27]. In other words, these numerical methods are usually incapable of seeking unstable solutions. It is difficult, therefore, to track the evolutions of periodic responses as a change of the solution stability accompanies very often with a bifurcation.

The rest of this paper is organized as follows. In Section 2, we will introduce the YA principle based on which the method is proposed. We will then solve linear oscillators subjected to a harmonically excitation. The fractional Duffing oscillator will be investigated in Section 3 by the HB method plus the arc-length continuation, namely the IHB method. We discuss the main analysis results in Section 4. To further extend the presented method, we solve limit cycle solutions of the fractional van der Pol oscillator in Section 5. This paper will end with some conclusions and remarks in Section 6.

2. Solution of linear fractional oscillator problem

The YA method was first proposed to solve a single degree-of-freedom spring-mass-damper system with a fractional damper [11]

$$mD^2x(t)+cD^{\alpha}x(t)+kx(t)=f\ \sin(\omega t) \tag{1}$$

where m, c and k represent the mass, damping coefficient, and stiffness, respectively; f and ω denote the amplitude and angular frequency of the external excitation, respectively. The term, $D^{\alpha}x(t)$ $(0<\alpha<1)$, is the Caputo derivative of order α

$$D^{\alpha}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{Dx(\sigma)}{(t-\sigma)^{\alpha}}d\sigma \tag{2}$$

where $\Gamma(\alpha)=\int_0^{\infty}e^{-z}z^{\alpha-1}dz$ represents the Gamma function. Using the relationship $1/\Gamma(1-\alpha)=\Gamma(\alpha)\sin\pi\alpha/\pi$, Yuan and Agrawal deduced $D^{\alpha}x(t)=\mu\int_0^{\infty}\phi(y,t)dy$ with $\mu=2\ \sin\pi\alpha/\pi$ and $\phi(y,t)=y^{2\alpha-1}\int_0^t e^{-(t-\tau)y^2}Dx(\tau)d\tau$. Eq. (1) can be rewritten as

$$mD^2x(t)+c\mu\int_0^{\infty}\varphi(y,t)dy+kx(t)=f\ \sin\omega t \tag{3}$$

By implementing the YA approach introduced in Appendix A, steady state responses (if there are any) can be obtained by numerically integrating Eq. (A3) in a duration long enough so that transient responses are damped out. In this section, we will apply the harmonic balance (HB) method to directly solve the periodic responses. Our procedures are based on the equivalence between system (1) and (3). When the transient responses are damped out and steady state responses are obtained, we can approximate the solution as

$$x(t)=\gamma\ \cos(\omega t)+\beta\ \sin(\omega t) \tag{4}$$

with the coefficients (γ and β) as unknown constants.

Using the equivalence between the fractional derivative and the parametric improper integral

$$D^{\alpha}x(t)=\mu\int_0^{\infty}y^{2\alpha-1}\left[\int_0^t e^{-(t-\tau)y^2}Dx(\tau)d\tau\right]dy \tag{5}$$

one can calculate

$$D^{\alpha}\ \cos\omega t=\mu Y_1\ \cos\omega t-\mu Y_2\ \sin\omega t-\mu\omega J_1(t) \tag{6}$$

$$D^{\alpha}\ \sin\omega t=\mu Y_2\ \cos\omega t+\mu Y_1\ \sin\omega t-\mu\omega^2 J_2(t) \tag{7}$$

with $Y_1=\omega^2\int_0^{\infty}\frac{y^{2\alpha-1}}{\omega^2+y^4}dy=\omega^{\alpha}\int_0^{\infty}\frac{y^{2\alpha-1}}{1+y^4}dy$, $Y_2=\omega\int_0^{\infty}\frac{y^{2\alpha+1}}{\omega^2+y^4}dy=\omega^{\alpha}\int_0^{\infty}\frac{y^{2\alpha+1}}{1+y^4}dy$, $J_1(t)=\int_0^{\infty}\frac{y^{2\alpha-1}e^{-ty^2}}{\omega^2+y^4}dy$, and $J_2(t)=\int_0^{\infty}\frac{y^{2\alpha+1}e^{-ty^2}}{\omega^2+y^4}dy$.

Notice that improper integrals, Y_i and J_i, are both convergent as long as ω is a positive constant for $0<\alpha<1$. Moreover, the non-periodic functions, J_i, are uniformly convergent to 0 as t increasing, as shown in Fig. 1. The calculations of Y_i and J_i are shown in Appendix B. Different from ordinary derivatives, the fractional derivatives of periodic functions are essentially non-periodic. As long as t is large enough, they can be approximated as periodic functions. The time history of $D^{0.5}\cos(t)$ shown in Fig. 2 approaches a periodic curve with t increasing.

Substituting of Eq. (4) into (3), neglecting the decaying terms $J_1(t)$ and $J_2(t)$ and setting coefficients corresponding to cosine and